WIGNER LECTURE 2013

THE HEURISTICALLY-BASED GENERALIZED PERTURBATION THEORY

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Foreword

This lecture consists of two parts:

- The first one corresponds to a short general introduction. In it the steps of the development of the HGPT methodology are briefly described together with the various fields of interest to which the method has been applied.

- In the second part a relevant application of this methodology is presented in more detail.

THE STEPS OF THE HGPT METHOD DEVELOPMENT

The interest on perturbation methods for reactor physics studies started during my stage as a young physicist associated to the Reactor Physics Division of the Argonne National Laboratory in years 1961-62.

At that time I was involved in fast reactors analysis work [1] and it was during this period that the Division Director Robert Avery posed to me the question on the possibility of developing perturbation methods for analysing functionals of the neutron flux, e.g., reaction rate ratios, in analogy with the methods used for calculating reactivity coefficients.

It was in later years that my research activity focused on this subject [2], while I was involved in research activities at the Casaccia Center of the Italian National Committee of Nuclear Energy (CNEN).

At that time, with the valid contribution of a strongly motivated group of researchers, at CNEN it was developed and applied to reactor physics analysis what became widely known as the Generalised Perturbation Theory (GPT) method.

The first relevant application of this method was presented in 1966 at the ANL Conference on Fast Critical Experiments and their Analysis [3].

The method used adopted heuristic concepts based on conservation principles, extending the method formerly proposed by L.N. Usachev [4] to any functional of the real and adjoint neutron fluxes, such as reactivity coefficients, effective prompt neutron lifetimes, effective delayed neutron fractions.

A number of codes implementing the GPT methodology were written at that time for the calculation of importance functions and sensitivity coefficients [3,5] associated to the responses to be analysed.

For a number of years these codes have been a reference for GPT sensitivity analysis applications.

This heuristic methodology, initially limited to the neutron domain, was then further extended to the nuclide field [6, 7] enabling, in particular, sensitivity studies relevant to the fuel depletion evolution.

One of the first, important applications of the HGPT methodology concerned the sensitivity analysis of integral data measured in critical facilities (ZPRs, MASURCA, etc.) for their use in differential and integral data adjustment exercises [8, 9, 10].

In further developments the HGPT methodology was applied:

- to nonlinear problems [11], in particular, to the coupled neutron/nuclide field for fuel cycle analysis;

- to the estimation of spatial shifts of power pick points following a perturbation [12];

- to reactor design optimization [13];

- to the development of the EGPT method [14] by which, for the analysis of reactivity coefficients, the calculation of the importance functions, implying the solution of inhomogeneous equations, is replaced by the calculation of functions, solutions of simpler homogeneous equations governed by modified operators;

- to the analysis of subcritical (ADS) reactors [15]. This led to the definition of 'generalized reactivity', properly taking into account the intensive control variable (for instance, the neutron source strength) required for maintaining the established power level. This 'generalized reactivity' appears, in particular, in the derivation of the point kinetic equations governing the normalized power of a subcritical system [16, 17];

- to the development of a method for sensitivity analysis in the reliability domain [18];

- to the development of a method by which the information obtained online through a system of neutron measuring devices such as self-powered neutron detectors (SPNDs) inserted in the core of a nuclear power reactor allows the online detection of a possible hot spot during plant operation [19, 20].

2. THE HGPT METHOD FOR ADS ANALYSIS

In this part of the lecture the formulation and the application of the HGPT methodology to the sensitivity analysis of subcritical (ADS) systems will be presented and discussed.

A similar subject was considered by Dan Cacuci in his Eugene Wigner Keynote Lecture at the PHYSOR 2002 International Meeting in Seoul [21]. The motivation of choosing this subject at that time seemed quite appropriate, considering the relevance ADS systems are assuming in perspective nuclear energy scenarios.

In the following, a few pertinent comments will be introduced on remarks made by Cacuci during his lecture.

In his intervention Cacuci refers to an approach presented in an article by Gandini and Salvatores [17], relevant to the point kinetics equation in ADS's and making use of the so called Heuristically based Generalised Perturbation Theory (HGPT).

He then emphasises that this, and other previous approaches, are improper for the analysis of an ADS since they adopt *"methods that are fundamentally based on the use of the eigenfunction expansions and perturbation theory ‘a la Wigner’ ... These authors postulate the existence of an appropriate homogeneous fictitious steady-state eigenvalue problem...”.*

And, moreover, *"... the recent method proposed by Gandini and Salvatores is based on a simplified, time-independent version"* of the adjoint equation *"that (i) neglects the delayed neutrons and (ii) uses an adjoint source which attempts to describe the equivalent number of fissions that would correspond to the accelerator beam proton 'steady-state' power. The specific form of this equation is*

$$-\mathbf{\Omega}\cdot\nabla n_o^* + \Sigma_t n_o^* = S^+ n_o^* + [\chi_P(1-\beta) + \chi_D]F^* n_o^* + \gamma\Sigma_f / W_o \qquad (1)$$

where γ *represents the energy released per fission and W would represent a 'steady-state power' of the sub-critical core.*"

I believe that in the course of this presentation these issues will appear appropriately clarified.

2.1. The HGPT approach

Since the beginning of nuclear reactor physics studies, perturbation theory has played an important role. As well known, it was first proposed by Wigner [22] to study fundamental quantities such as the reactivity worths of different materials in the reactor core.

It is also well known that this first formulation, today widely used for reactor analysis, makes a consistent use of the classical adjoint flux concept.

The HGPT approach, fundamentally based, as we shall see, on the ‘importance conservation principle’, is generally intended to be used for defining perturbation expressions relevant to a variety of responses in stationary, as well as time dependent, linear and nonlinear fields.

Compared with direct calculations, it allows to obtain with relatively limited effort the sensitivity coefficients of the responses of interest, and may then be very well used, as it has been extensively done, in a variety of studies.

The concept of importance was first defined by B.B. Kadomtzev [23], in the radiation transport field, and then considered by L.N. Usachev [4], as the contribution to a response in a critical system by a neutron through its progeny, and by J. Lewins [24], as interpretation of the adjoint function adopted in variational techniques.

It is quite clear that the HGPT methodology is not based on "*eigenfunctions expansions a la Wigner*".

To be also reminded that the HGPT approach and the variational one [24, 25] differ only in the procedure for arriving at the sensitivity/perturbation expressions, these resulting equivalent to each other as demonstrated for different fields by Greenspan [26] and Marques Alvim et al. [27].

The preference, or the merit of each method for arriving at the perturbation/sensitivity expressions of interest is beyond the scope of this lecture.

We limit to note that, whereas with the variational approach one makes use of the mathematical concept of adjoint function and of its properties, and then realizes that it may be associated with the concept of importance, with the HGPT approach one starts with defining this latter quantity and heuristically arrives at the equation governing it and, finally, at the sensitivity/perturbation expressions.

The importance may be subsequently associated with the adjoint function, while the 'source reciprocity relationship', which will be defined in the following, may be associated with its properties.

From these arguments it has been thought appropriate to refer to the method described in this lecture with the acronym HGPT (Heuristically Based Generalized Perturbation Theory), in order to evidence the approach that was originally considered.

2.1.1. The importance function

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle.

The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones.

For all these fields the equation governing the importance function was obtained directly basing on the so called ‘importance conservation principle’, i.e., by imposing that on average the contribution to the chosen response from a particle introduced at a given time in a given phase space point of the system is conserved through time, directly or trough its progeny.

Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.

Consider a linear particle field density represented by vector $\mathbf{f}$ (e.g., the multigroup neutron density field) and a response Q of the type

$$Q = \int_{t_o}^{t_F} < \mathbf{s}^+, \mathbf{f} > dt \equiv << \mathbf{s}^+, \mathbf{f} >> , \qquad (2)$$

where $\mathbf{s}^+$ is an assigned vector function and where < > indicate integration over the phase space.

Weighting all the particles inserted into the system, let's assume produced by a source $\mathbf{s}$, with the corresponding importance ($\mathbf{f}^*$) will obviously give the response itself, i.e.,

$$<<\mathbf{f}^*, \mathbf{s}>> = Q = <<\mathbf{s}^+, \mathbf{f}>> , \qquad (3)$$

which represents the so called 'source reciprocity relationship'.

From the first derivations mentioned above the rules for determining the equation governing the importance function $\mathbf{f}^*$ were learned. They imply, with respect to the equation governing the real function $\mathbf{f}$:

- change of sign of the odd derivatives,
- transposing matrix elements,
- reversing the order of operators,
- substitution of the real source **s** with $\mathbf{s}^+$.

The first three rules will be generally called "operator reversal" rules.

The HGPT method was then extended to any field governed by linear operators for which the rules for their reversal were known. In particular, it was extended to the derivative fields, obtained by expanding to first order, around a given starting solution, a number of important nonlinear equations, as those governing:

- the coupled neutron/nuclide field, relevant to core evolution and control problems,
- the temperature field, relevant to thermohydraulics

2.1.2. General formulation

Consider a generic physical system defined by a number of parameters p_j and described by an N-component vector field $\mathbf{f}$ obeying equation

$$\mathbf{m}(\mathbf{f}|\mathbf{p}) = \mathbf{0} \quad . \tag{4}$$

Vector $\mathbf{f}(\mathbf{q},t)$ generally depends on the phase space coordinates $\mathbf{q}$ and time t. Vector $\mathbf{p}$ represents the set of independent parameters p_j fully describing the system.

Equation (4) can be viewed as an equation comprising linear, as well as nonlinear, operators and is assumed to be derivable with respect to parameters p_j and (in the Frechet sense) component functions f_n .

Consider now a response of interest, or functional Q, as expressed by Eq.(2):

$$Q = \int_{t_o}^{t_F} < \mathbf{s}^+, \mathbf{f} > dt \equiv << \mathbf{s}^+, \mathbf{f} >> ,$$

In the following, we shall look for an expression giving perturbatively the change δQ of the response Q in terms of perturbations δp_j of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to each parameter p_j would be obtained.

Expanding equation (4) to first order around a reference solution, we may write

$$\sum_{j=1}^{J} \delta p_j (H\mathbf{f}_{/j} + m_{/j}) = \mathbf{0} \quad , \qquad (5)$$

where $\mathbf{f}_{/j}\ (\equiv \frac{d\mathbf{f}}{dp_j})$ are derivative functions, $\mathbf{m}_{/j}\ (\equiv \frac{\partial \mathbf{m}}{\partial p_j})$ a source term and H the linear operator

$$H = \frac{\bar{\partial}\mathbf{m}}{\partial \mathbf{f}} \equiv \begin{vmatrix} \frac{\bar{\partial} m_1}{\partial f_1} & \cdots & \frac{\bar{\partial} m_1}{\partial f_N} \\ \vdots & \vdots & \vdots \\ \frac{\bar{\partial} m_N}{\partial f_1} & \cdots & \frac{\bar{\partial} m_N}{\partial f_N} \end{vmatrix} .$$

By $\frac{\bar{\partial}}{\partial f_n}$ we have indicated Frechet derivatives [28].

Since parameters p_j, and then their changes δp_j, have been assumed to be independent from each other, it must follow

$$H\mathbf{f}_{/j} + \mathbf{m}_{/j} = \mathbf{0} \;, \qquad (6)$$

which represents the (linear) equation governing the derivative functions $\mathbf{f}_{/j}$.

Consider now the functional, dependent on the derivative function $\mathbf{f}_{/j}$,

$$Q_j = \ll \mathbf{h}^+, \mathbf{f}_{/j} \gg \ . \qquad (7)$$

Introducing the importance ($\mathbf{f}^*$) associated with this functional, and recalling the ‘source reciprocity relationship’, we may write:

$$Q_j = \ll \mathbf{f}^*, \mathbf{m}_{/j} \gg \qquad (8)$$

where the importance $\mathbf{f}^*$ obeys the linear, index-independent equation

$$H^*\mathbf{f}^* + \mathbf{h}^+ = \mathbf{0} \ , \qquad (9)$$

H^* being obtained by reversing operator H.

We can easily see that the sensitivity s_j of functional Q with respect to a system parameter p_j can be written

$$s_j \equiv \frac{dQ}{dp_j} = \ll \frac{\partial \mathbf{h}^+}{\partial p_j}, \mathbf{f} \gg + \ll \mathbf{f}^*, \mathbf{m}_{/j} \gg \qquad (10)$$

where the first term at the right-hand side represents the so called, easy to calculate, direct term.

The overall change δQ due to perturbations δp_j $(j=1,2,\ldots,J)$ of system parameters can be written, to first order,

$$\delta Q = \sum_{j=1}^{J} \delta p_j \left[\ll \frac{\partial \mathbf{h}^+}{\partial p_j}, \mathbf{f} \gg + \ll \mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_j} \gg \right]. \qquad (11)$$

2.2. The HGPT applied to subcritical system analysis

Due to its generality, the HGPT approach may be applied without ambiguity to the analysis of critical, as well as subcritical systems.

Indications on this latter possibility were advanced since the late 60's [29, 30, 31, 32].

Inherent with the HGPT theory is the concept of the control to be associated with a system generally subject to a constraint on the power, or flux level.

In his keynote lecture at the PHYSOR 2002 Meeting Cacuci says: *"Classical perturbation theory certainly cannot be used for optimal operation and control of an ADS".* But, while this is true in relation to the classical perturbation theory, *'a la Wigner',* in which the control role is fictitiously played by the coefficient (λ) multiplying the fission source, the same thing cannot be said with respect to the HGPT theory which may account for the real control (as a control rod insertion, or the intensity of an extraneous neutron source strength) [33, 34].

Introducing the general frame of optimal control theory does not change this conclusion. As well known, control theory intrinsically uses the sensitivity coefficients of responses of interest, among which the target quantity, during the various steps of an optimization search [35]. The issue here is that of adopting a correct, unbiased sensitivity theory accounting for the *real* control adopted, in a critical, as well as in a subcritical reactor system. Which is just what the HGPT theory is doing.

The above concept of control-related HGPT theory in relation to subcritical systems has been considered in some depth since 1997 [15, 16].

Special attention has been given to a response represented by the very control variable, in particular, the source strength itself. This gives rise to a peculiar perturbation expression by which it is possible to evaluate control changes (for instance, at the end of reactor cycle life, in an evolution study) following a perturbation of any system parameter.

2.2.1. The HGPT methodology at quasi-static conditions

The methodology relevant to long term nuclide/neutron core evolution analysis may be very well applied to source driven, subcritical systems.
Consider the equations governing the neutron density $\mathbf{n}$, the nuclide density $\mathbf{c}$ and the (intensive) control function ρ(t) during the core evolution (burn-up), in the form

$$\mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho\,|\,\mathbf{p}) = -\frac{\partial \mathbf{n}}{\partial t} + B\mathbf{n} + \rho\mathbf{s}_n = 0 \qquad (12)$$

$$\mathbf{m}_{(c)}(\mathbf{n},\mathbf{c}\,|\,\mathbf{p}) = -\frac{\partial \mathbf{c}}{\partial t} + E\mathbf{c} + \mathbf{s}_c = 0 \qquad (13)$$

$$m_{(\rho)}(\mathbf{n},\mathbf{c}\,|\,\mathbf{p}) = <\mathbf{c}, S\mathbf{n}> - \; W = 0 \qquad (14)$$

where B and E depend on fuel and neutron densities $\mathbf{c}$ and $\mathbf{n}$, respectively. At quasi-static conditions, the time derivative at right side of Eq.(12) may be neglected during a calculation procedure.

Any response, functional of variables $\mathbf{n}$, $\mathbf{c}$, and ρ, could be considered for analysis. We think instructive to limit here consideration to the response defined by the expression

$$Q = \rho(t_F) \equiv \int_{t_o}^{t_F} \delta(t - t_F)\rho(t)dt \qquad (15)$$

which corresponds to the relative source strength required at t_F for maintaining the imposed power level.

We may assume that, at unperturbed conditions, $\rho(t)=1$ in the interval (t_o,t_F). If some system parameter (for instance, the initial enrichment, or some other material density) is altered, as in an optimization search analysis, it may be of interest to evaluate the corresponding change of ρ at the end of cycle, to make sure that given upper limit specifications of the source strength are not exceeded.

Along with the HGPT methodology, the equations for the corresponding importance functions result

$$-\frac{\partial \mathbf{n}^*}{\partial t} = B^*\mathbf{n}^* + \Omega_c^*\mathbf{c}^* + S^T\mathbf{c}\rho^* \qquad (16)$$

$$-\frac{\partial \mathbf{c}^*}{\partial t} = E^*\mathbf{n}^* + \Omega_n^*\mathbf{n}^* + S\mathbf{n}\rho^* \qquad (17)$$

$$<\mathbf{n}^*, \mathbf{s}_n> + \ \delta(t-t_F) \ = \ 0 \qquad (18)$$

Ω_c^* and Ω_n^* being coupling operators, adjoint of operators

$\Omega_c \left[= \frac{\bar{\partial}(E\mathbf{c})}{\partial \mathbf{n}}\right]$ and $\Omega_n \left[= \frac{\bar{\partial}(B\mathbf{n})}{\partial \mathbf{c}}\right]$.

Eq. (18) corresponds to an orthonormal condition for $\mathbf{n}^*$.

If we define vectors

$$\mathbf{f} = \begin{vmatrix} \mathbf{n} \\ \mathbf{c} \\ \rho \end{vmatrix}, \tag{19}$$

$$\mathbf{f}^* = \begin{vmatrix} \mathbf{n}^* \\ \mathbf{c}^* \\ \rho^* \end{vmatrix} \tag{20}$$

$$\mathbf{m}(\mathbf{n},\mathbf{c},\rho \,|\, \mathbf{p}) = \begin{vmatrix} \mathbf{m}_{(n)}(\mathbf{n},\mathbf{c},\rho \,|\, \mathbf{p}) \\ \mathbf{m}_{(c)}(\mathbf{n},\mathbf{c} \,|\, \mathbf{p}) \\ m_{(\rho)}(\mathbf{n},\mathbf{c} \,|\, \mathbf{p}) \end{vmatrix} \tag{21}$$

the sensitivity coefficient s_k of functional Q with respect to parameter p_k will then be given by the expression

$$s_k \equiv \frac{d\rho(t_F)}{dp_k} = \ll \mathbf{f}^*, \frac{\partial \mathbf{m}}{\partial p_k} \gg . \tag{22}$$

In an explicit form, we may write

$$\frac{d\rho(t_F)}{dp_k} = [<\mathbf{n}_F^*\frac{\partial}{\partial p_k}(B\mathbf{n}+\mathbf{s}_n)> + \rho_F^*\frac{\partial}{\partial p_k}(<\mathbf{c}, S\mathbf{n}> - W)]_{t_F}$$

$$+\int_{t_o}^{t_F}[<\mathbf{n}^*, \frac{\partial}{\partial p_k}(B\mathbf{n}+\mathbf{s}_n) + <\mathbf{c}^*, \frac{\partial E}{\partial p_k}\mathbf{c}> + \rho^*\frac{\partial}{\partial p_k}(<\mathbf{c}, S\mathbf{n}> - W)]dt$$

(23)

where $\mathbf{n}_F^*$ (value of $\mathbf{n}^*$ at t_F) obeys equation

$$B^*\mathbf{n}_F^* + \rho_F^* S^T\mathbf{c}(t_F) = 0 \tag{24}$$

and

$$\rho_F^* = -\frac{1}{W} \quad . \tag{25}$$

2.2.2. Stationary Case

To study a given subcritical system at stationary conditions (as is the case at the beginning of its cycle life), we may consider the same system above in which the neutron source and the nuclide density are assumed time-independent.

Their governing equations can then be written, in case the power level is controlled by the source strength,

$$B\mathbf{n}_o + \rho_o\, \mathbf{s}_{n,o} = 0 \qquad (26)$$

$$<\mathbf{c}_o, S\,\mathbf{n}_o> - W_o = 0\,. \qquad (27)$$

Also here we shall assume that at unperturbed conditions $\rho_o = 1$.

The sensitivity coefficient of the response ρ_o in this case will result

$$\frac{\partial \rho_o}{\partial p_k} = \rho_o^* [< \mathbf{n}_o^*, \frac{\partial}{\partial p_k}(B\mathbf{n}_o + \mathbf{s}_{n,o}) > + \frac{\partial}{\partial p_k}(< \mathbf{c}_o, S\mathbf{n}_o > - W_o)] \quad (28)$$

where

$$\rho_o^* = -\frac{1}{W_o} \quad (29)$$

and $\mathbf{n}_o^*$ obeys equation

$$B^* \mathbf{n}_o^* + \rho_o^* S^T \mathbf{c}_o = 0 \quad . \quad (30)$$

If, rather than via the source strength, the power level reset control is assumed to be regulated via neutron absorption, so that the control ρ_o would enter into operator B, the sensitivity coefficient would be given always by Eq. (28), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, \frac{\partial B}{\partial \rho} \mathbf{n} >} . \qquad (31)$$

We might as well consider a (fictitious) control parameter (λ) multiplying the fission source $F\mathbf{n_o}$.

The sensitivity coefficient would be given again by Eq. (28), but with

$$\rho_o^* = -\frac{1}{< \mathbf{n}_o^*, F\mathbf{n}_o >} . \qquad (32)$$

2.2.3. Reactivity of Subcritical Systems

For resetting the power level, we have considered above different control mechanisms to which the following types of equations governing the neutron density may be associated:

$$B(\mathbf{p})\mathbf{n}_o + \rho_o \mathbf{s}_{n,o}(\mathbf{p}) = 0 \qquad \text{(source control)} \qquad (33)$$

$$B(\rho_o|\mathbf{p})\mathbf{n}_o + \mathbf{s}_{n,o}(\mathbf{p}) = 0$$

(neutron absorption, or fission control) (34)

These equations may be generally represented by vector

$$\mathbf{m}_{(n,o)}(\mathbf{n}_o, \rho_o | \mathbf{p}) = 0 \,. \qquad (35)$$

The sensitivity expression (28) may be generalized so that

$$\frac{d\rho_o}{dp_j} = -\frac{< \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial p_j} > + \dfrac{\partial}{\partial p_j}(< \mathbf{c}_o, S\mathbf{n}_o > - W_o)}{< \mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o} >} \,, \qquad (36)$$

with $\mathbf{n}_o^*$ obeying Eq. (30).

Assuming that the power W_o appearing in Eq. (36) is not subject to perturbation, we may write:

$$\delta\rho_o = -\frac{<\mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)}> + <\mathbf{n}_o, \delta(S^T\mathbf{c}_o)>}{<\mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o}>}, \quad (37)$$

where

$$\delta\mathbf{m}_{(n,o)} = \sum_j \delta p_j \frac{\partial\mathbf{m}_{(n,o)}}{\partial p_j} \quad (38)$$

$$\delta(S^T\mathbf{c}_o) = \sum_j \delta p_j \frac{\partial(S^T\mathbf{c}_o)}{\partial p_j}. \quad (39)$$

We may as well say that the perturbation $\delta\mathbf{m}_{(n,o)}$ [and $\delta(S^T\mathbf{c}_o)$] would produce a power level change equivalent to that produced by a change of the control variable ρ.

Indicating this control change as δK_ρ , we may write

$$\delta K_\rho = \frac{<\mathbf{n}_o^*, \delta\mathbf{m}_{(n,o)}> + <\mathbf{n}_o, \delta(S^T\mathbf{c}_o)>}{<\mathbf{n}_o^*, \dfrac{\partial\mathbf{m}_{(n,o)}}{\partial\rho_o}>} . \qquad (40)$$

In the case of a (fictitious) control on the neutron fission source, setting λ in place of ρ to distinguish this peculiar case, we may explicitly write the expression for the corresponding control variable change

$$\delta K_\lambda = \frac{< \mathbf{n}_o^*, \delta B \mathbf{n}_o >}{< \mathbf{n}_o^*, F \mathbf{n}_o >} + \frac{< \mathbf{n}_o^*, \delta \mathbf{s}_{n,o} >}{< \mathbf{n}_o^*, F \mathbf{n}_o >} + \frac{< \mathbf{n}_o, \delta(S^T \mathbf{c}_o) >}{< \mathbf{n}_o^*, F \mathbf{n}_o >} . \quad (41)$$

The first term at the right side closely resembles the reactivity expression for critical systems. So, we shall call the quantity δK_λ a 'generalized reactivity'.

The second term at the right hand side may be defined as the "source reactivity", whereas the last one corresponds to a "direct effect".

The first term at right hand side can be demonstrated to formally approach the standard reactivity expression as the system approaches criticality conditions [15].

To account for a generic ρ-mode control mechanism, we shall extend this definition to δK_ρ, similarly defined by Eq. (40), i.e.,

$$\delta K_\rho = \frac{<\mathbf{n}_o^*, \delta B \mathbf{n}_o>}{<\mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o}>} + \frac{<\mathbf{n}_o^*, \delta \mathbf{s}_{n,o}>}{<\mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o}>} + \frac{<\mathbf{n}_o, \delta(S^T \mathbf{c}_o)>}{<\mathbf{n}_o^*, \dfrac{\partial \mathbf{m}_{(n,o)}}{\partial \rho_o}>} . \tag{42}$$

and call it generalized ρ-mode reactivity.

2.2.4. Point Kinetics

Let us now consider equations governing the neutron flux ϕ ($\equiv \mathrm{V}\mathbf{n}$) and precursor m_i (i=1,2,...,I) in a multigroup (G groups) neutron energy scheme:

$$\mathrm{V}^{-1}\frac{d\phi}{dt} = \mathrm{A}\phi + (1-\beta)\chi_P S_f^G \phi + \chi_D \mathbf{u}\sum_{i=1}^{I}\lambda_i m_i + \mathbf{s}_n \qquad (43)$$

$$\frac{dm_i}{dt} = \beta_i \nu \boldsymbol{\Sigma}_f^T \phi - \lambda_i m_i \qquad (44)$$

where A is the transport, capture and scattering matrix operator, V the diagonal neutron velocity matrix, $\mathbf{u}$ is a unit (G component) vector and

$$S_f^X = \begin{vmatrix} \nu\Sigma_{f,1} & \ldots & \nu\Sigma_{f,G} \\ \ldots & \ldots & \ldots \\ \nu\Sigma_{f,1} & \ldots & \nu\Sigma_{f,G} \end{vmatrix}_{(X\ \text{rows})}, \qquad (45)$$

$$\boldsymbol{\Sigma}_f^T = \begin{vmatrix} \Sigma_{f,1} & \ldots & \Sigma_{f,G} \end{vmatrix}, \qquad \chi_z = \mathrm{diag}\begin{vmatrix} \chi_{z,1} & \ldots & \chi_{z,G} \end{vmatrix} \qquad (46)$$

Consider at initial state conditions the neutron importance $\mathbf{n}_o^*$ and the corresponding precursor density importance $\mathbf{m}_o^*$ associated with the normalized power P_o. The importance $\mathbf{n}_o^*$ results governed by the equation

$$A_o^* \mathbf{n}_o^* + \nu S_{f,o}^T \left[(1-\beta)\chi_P + \beta \chi_D \right] \mathbf{n}_o^* + \frac{\gamma}{W_o} \boldsymbol{\Sigma}_{f,o} = 0 \qquad (47)$$

while, by the very definition of importance, the precursor importance results given by the expression

$$m_{i,o}^* \equiv m_o^* = \mathbf{u}^T \chi_D \mathbf{n}_o^* . \qquad (48)$$

Multiplying on the left Eqs. (43) and (44), relevant to the real neutron and precursor densities, by their importances $\mathbf{n}_o^*$ and $\mathbf{m}_o^*$, respectively, space-integrating and dropping second order terms, after some manipulations we obtain the equations governing the normalized (relative) power P ($\equiv$ W/W$_o$) and the 'effective' precursor densities ξ_i .

$$l_{eff}\frac{dP}{dt} = (\rho_{gen} - \alpha\beta)P + \alpha\sum_{i=1}^{I}\lambda_i\xi_i + \zeta(1-P) + \rho_{source} \qquad (49)$$

$$\frac{d\xi_i}{dt} = \beta_i P - \lambda_i\xi_i \qquad (50)$$

Definitions of quantities which appear in Eqs. (49) and (50):

$$\xi_i = \zeta < m^*_{s,o} m_i > \qquad \text{(effective precursor density)}$$

$$l_{eff} = \zeta < \mathbf{n}^*_{s,o}, V^{-1}\phi_o > \qquad \text{(effective prompt neutron lifetime)}$$

$$\rho_{gen} = \zeta\left(< \mathbf{n}^*_{s,o}, \{ \delta A + (1-\beta)\chi_P + \beta\chi_D]\delta S_f \}\phi_o > + \frac{\gamma}{W_o} < \delta\Sigma_f, \phi_o \right) \qquad \text{(generalised reactivity)}$$

$$\rho_{source} = \zeta < \mathbf{n}^*_{s,o}, \delta\mathbf{s}_n > \qquad \text{(source reactivity)}$$

$$\frac{1}{\zeta} = \overline{< \mathbf{n}^*_{s,o}, \chi S_f \phi >} \equiv (1-\beta) < \mathbf{n}^*_{s,o}, \chi_P S_f \phi > + \beta < \mathbf{n}^*_{s,o}, \chi_D S_f \phi$$

$$\alpha = \zeta < \mathbf{n}^*_{s,o}, \chi_D S_{f,o} \phi_o >$$

Note. Eqs. (49) and (50) may be considered a generalization of the point kinetic equation derived by Usachev for critical systems [L.N. Usachev, 1st ICPUE-UN, 5, 503 (1955). It may be demonstrated that they converge with his equation when the system approaces criticality.

.

To note that quantity ζ plays the role of an appropriate measure of the system subcriticality.

To show this, consider first the two subcriticality measures generally adopted, K_{eff} and K_{source} .

K_{eff} is defined as

$$K_{eff} = \frac{\overline{<\phi_o^*, \chi S_f \phi_o>}}{<\phi_o^*, \mathbf{s}_{n,o}> + \overline{<\phi_o^*, \chi S_f \phi_o>}} , \qquad (51)$$

with ϕ_o^* the standard adjoint flux, associated with the fundamental mode.

This measure appears justified for studies at conditions not too far from criticality.

K_{source} is defined as a multiplication factor implying a real neutron flux in presence of an external source, and then formed by a superposition of eigenfunctions.

It does not take into account the importances of fission and source neutrons with respect to the power.

It is given by the expression

$$K_{source} = \frac{\overline{< \mathbf{u}, \chi S_f \phi >}}{< \mathbf{u}, \mathbf{s}_{n,o} > + \overline{< \mathbf{u}, \chi S_f \phi >}} \qquad (52)$$

where $\mathbf{u}$ is a unit vector.

So, taking into account the importances of fission and source neutrons, and recalling that $<\mathbf{n}_o^* s_{n,o}>=1$, me may define the multiplication coefficient

$$K_{sub} = \frac{\overline{<\mathbf{n}_{s,o}^*, \chi S_f \phi>}}{<1+\overline{<\mathbf{n}_{s,o}^*, \chi S_f \phi>}} \equiv \frac{\frac{1}{\zeta}}{1+\frac{1}{\zeta}} . \qquad (53)$$

Quantity ζ then may be written as

$$\zeta = \frac{1-K_{sub}}{K_{sub}} , \qquad (54)$$

and may be clearly taken as a consistent measure of the distance of the system from criticality.

A relevant example of application of the HGPT methodology in the kinetic domain is illustrated in a recent article by Dos Santos et al. [36] in which they propose a method for determining with precision the reactivity levels of subcritical systems making use of the point kinetic equations described above.

The method is based only on measured quantities such as counting rates of the detectors employed and the parameters arising from the least squares fitting of the APSD (Auto Power Spectral Density).

An important aspect of their proposed method is that detector efficiencies, quantities required in other procedures such as the Neutron Source Multiplication (NSM) method, do not result needed with their approach.

Their method was applied to measure the reactivity of several subcritical configurations of the IPEN/MB-01 reactor.

Measurements were performed at several degrees of subcriticality (up to around -7000 pcm). The APSD data were least squares fitted to get the prompt decay mode (α) and other quantities.

The final measurements resulted of very good quality.

The proposed experimental method appears to show clearly that the classical point kinetic theory does not describe correctly the measured reactivity.

Instead, the reactivity inferred from this model follows closely the subcriticality index ζ, Eq.(54), for the source arrangements in the experiment.

FINAL NOTE

As a final note in this ‘Wigner lecture’, I like to remind the words written by Jeffery Lewins and Martin Becker in the preface of volume 19 of the annual series “Advances in Nuclear Science and Technology”, published in 1987 as a festschrift in honor of Eugene P. Wigner.

They, referring to the work I contributed in that publication, say:

“… We have already mentioned the introduction of variational methods or more particularly non-self adjoint perturbation methods in nuclear engineering by Eugene Wigner. It is appropriate therefore to include in this volume of homage a general treatise on the generalized perturbation method that has sprung from the seed Wigner sowed …”.

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