ON THE PHYSICS OF SUBCRITICAL SYSTEMS

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1. Introduction

The main lesson learnt so far in the four decades of pacific uses of atomic energy, after the two Geneva Conferences of 1955 and 1964, is that the accidents associated with nuclear energy posing major risk problems, i.e., greater than those perceived acceptable as an inevitable price to be paid for progress, are related to human factors. Such has been in fact the case with the events occurred at Three Mile Island, Chernobyl, and, more recently, Tokaimura. This latter accident, in particular, has broadened the perception of major possible risks outside the reactor system. Besides, although the quite positive experience with LWR reactors in the last decades cannot be forgotten, it is also perceived that a broader, world-wide nuclear energy expansion would pose a number of concerns related to the safety of reactor plants (power excursions, residual heat risk), as well as to those associated with the fuel flow (criticality accidents, fuel diversion, radiological risk, proliferation) and to the problems related to reprocessing (antiproliferation issues, fuel transportation risks). To find an answer to these issues in a long term scenario, the following objectives should be pursued:

1. A nuclear energy reactor concept assuring optimal characteristics in relation to economy, safety, anti-proliferation, anti-diversion, radiological risk minimization, public acceptance;
2. A viable, smooth transition from today’s nuclear energy production structure into that, at equilibrium, relevant to the new concept;
3. The possibility of closing the fission cycle, whenever a different important energy source is developed as a substitution, with minimal radio-toxicity from residual waste.

To answer these issues, a variety of different projects have been proposed in these years, among which we remind the molten salt reactor, the pebble-bed reactor and the encapsulated reactor (stemming from the well known IFR concept). Many of these systems are considered also in a subcritical configuration, i.e., as hybrid (ADS) concepts. A number of advantages (at short as well as long range terms) are claimed for this choice:

- an high degree of safety
- ability of significantly mitigating the waste stream
- ability to efficiently reducing the existing stocks of plutonium
- optimal use of uranium and thorium natural resources
- closure of the fission cycle.

The main justification for using ADS, rather than critical systems, appears however related to safety considerations, the distance from criticality conditions resulting equivalent, as will be shown, to an extra amount of delayed neutrons. This property, in particular, allows to consider them the best candidates as minor actinide (Am, Cu) incinerators, in consideration of the relatively small delayed neutron fraction associated with these elements.
In order to show the physical characteristics of these systems, we shall first describe the coupling of the reactor power with the accelerator. We shall then present a simple approach extending to ADS systems the reactivity balance approach for the fast reactor studies. Then, the concept of generalized reactivity applied to these systems will be illustrated, together with some peculiarities of their physical behavior.

2. Reactor/Accelerator Coupling

The coupling of a subcritical reactor with a proton accelerator producing spallation neutrons in a target implies an efficiency loss, increasing with the increasing subcriticality. To illustrate this concept, let us consider for simplicity an homogeneous subcritical bare system in a one group approximation driven by a source of intensity $S(r)$. The general solution of the flux may be written (Glasstone and Edlund, 1952), at asymptotic conditions,

$$\phi(r, t) = \frac{p}{K_c \Sigma_c} \sum_{n=1}^{\infty} \left[ \frac{K_n S_n}{(1-K_n)} \right] f_n(r). \quad (2.1)$$

where $f_n$ are the eigenfunctions, solutions of the equation

$$\nabla^2 f_n + B_n^2 f_n = 0,$$

$B_n^2$ being the geometrical buckling, $\Sigma_c$ is the macroscopic capture cross section of the core, $p$ the resonance escape probability, $K_n$ the multiplication coefficient associated with the $n$-th eigenfunction and $S_n$ are the moments of the neutron source expansion. We assume for simplicity a source distribution corresponding to the first flux eigenfunction, or fundamental mode (so that moments $S_n$ are zero for $n>1$). Multiplying by $\nu \Sigma_f$ and integrating over the whole core, we obtain the fission neutron source:

$$\int_{\text{core}} \phi(r) \nu \Sigma_f \, dr = \frac{p \nu \Sigma_f}{K_c \Sigma_a} \frac{K_{\text{eff}}}{(1-K_{\text{eff}})} S_1 \int_{\text{core}} f(r) \, dr = \frac{K_{\text{eff}}}{(1-K_{\text{eff}})} \int_{\text{core}} S(r) \, dr, \quad (2.2)$$

where $K_{\text{eff}} (\equiv K_1)$ is the fundamental eigenvalue.

Expression (2.2) may be extended to any geometry and neutron energy group representation, so that we may write:

$$\text{overall fission neutron source} = \int_{\text{core}} dE \int_{\text{core}} \phi(r, E) \nu \Sigma_f \, dr = \frac{K_{\text{eff}}}{(1-K_{\text{eff}})} \int_{\text{core}} dE \int_{\text{core}} S(r, E) \, dr \equiv \frac{sK_{\text{eff}}}{(1-K_{\text{eff}})}, \quad (2.3)$$

where

$$s = \int_{\text{core}} dE \int_{\text{core}} S(r, E) \, dr \quad (2.4)$$

is the overall extraneous neutron source.
Let us derive in the following a simple expression coupling the reactor power with that of the accelerator needed to produce an assigned reactor power, evidencing the dependence of this latter quantity from the subcriticality level and from other parameter associated with the accelerator.

Let's consider the accelerator power (in MW):

$$W_{\text{acc}} = cE_p/f_b,$$  \hspace{1cm} (2.5)

where $c$ represents the current (in mA), $E_p$ the proton energy (in GeV) and $f_b$ the accelerator efficiency.

The neutron source results

$$s = n_{mA}c = \frac{n_{mA}f_bW_{\text{acc}}}{E_p},$$

$n_{mA}$ representing the effective number of source neutrons produced per mA current.

The overall fission source (2.3) may then be written:

$$\int \int_{\text{core}} \phi(r,E)\Sigma_f dr \equiv V_{\text{core}} \phi \Sigma_f = \frac{n_{mA}f_bW_{\text{acc}}K_{\text{eff}}}{E_p(1 - K_{\text{eff}})}$$ \hspace{1cm} (2.6)

The quantity $n_{mA}$ is an increasing function of the proton energy $E_p$.

A current of 1 mA corresponds to 0.625x10\(^{16}\) protons/sec. We shall denote as $p_{mA}$ this quantity. Indicating with "m" the number of neutrons produced by each proton hitting the target (for example, of tungsten, or lead\(^1\), and assuming a weight "g" for these neutrons\(^2\), we have $n_{mA} = gmp_{mA}$.

The ratio $W_{\text{acc}}/W_e$ may be seen as the fraction of electricity lost in a plant of electric power $W_e (=\xi W_i)$. Substituting in (2.6) $V_{\text{core}} \phi \Sigma_f$ with $\nabla W_i / \kappa$, where $\kappa$ represents the units of energy (in MJ) per fission, it results:

$$\frac{W_{\text{acc}}}{W_i} = \frac{\nabla E_p(1 - K_{\text{eff}})}{K_{\text{eff}} f_b g m p_{mA}}$$ \hspace{1cm} (2.8)

i.e.,

$$\frac{W_{\text{acc}}}{W_e} = \frac{\nabla E_p(1 - K_{\text{eff}})}{K_{\text{eff}} f_b g m p_{mA}}.$$ \hspace{1cm} (2.9)

Recalling equation (2.5), we may also write the expression relevant to the current intensity needed for an ADS system of assigned power and subcriticality level:

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\(^1\) The number of spallation neutrons (m) produced by a proton hitting a lead target with energy $E_p$, in the energy range of interest, increases along the empirical expression (Andriamonje, et al., 1995):

$$m = 3.717 \times 10^{-5} E_p^2 + 3.396 \times 10^{-3} E_p - 0.367.$$

Recent indications give different values, lower by 10÷20%, for 1 GeV protons.

\(^2\) So, to account of their importance (in relation to the system power) with respect to the average one of fission neutrons.
\[ c = W_i \sqrt{1 - K_{\text{eff}}}/K_{\text{eff}} \kappa g m p_{/mA} \quad . \]  
(2.10)

The power of the proton beam will be

\[ W_{\text{beam}} = E_p W_i \sqrt{1 - K_{\text{eff}}}/K_{\text{eff}} \kappa g m p_{/mA} \quad . \]  
(2.11)

It is generally assumed that the optimal proton energy \( E_p \) is of the order of 1 GeV. This is mainly suggested by the need of limiting as much as possible the demaging of the window through which the proton beam accesses the multiplying region.\(^3\)

Then, assuming \( k_{\text{eff}} = 0.96, f_i = 0.5, f_e = 0.38, \nu = 2.7, m = 33 \text{ e g} = 1.2 \) and recalling that \( \kappa \) (energy units per fission) is of the order of 200 Mev (= \( 3.2 \times 10^{-17} \text{ MJ} \)), we obtain:

\[ W_{\text{acc}}/W_{e} = 0.074, \quad W_{\text{acc}}/W_{i} = 0.02, \quad c = 0.014 W_{i} \quad . \]

In this case, 7.4% of the electric power is absorbed by the accelerator. For a thermal power of 840 MW (corresponding to the dimensions of a PRISM like reactor), for a multiplication coefficient \( K_{\text{eff}} = 0.96 \), the power absorbed results \( W_{\text{acc}} = 24 \text{ MW} \), corresponding to proton current of about 10 mA.

Since \( \kappa = (\text{MJ})/\text{fission} \), it is also, denoting with "e" the elementary electron charge (in coulomb)

\[ 10^3 \kappa p_{/mA} = \frac{\kappa}{\text{e/sec}} = \frac{\kappa}{\text{e}} = \frac{(\text{MJ})/\text{fission}}{(\text{MeV})/\text{MeV}} = (\text{MeV})/\text{fission} = \varepsilon_f \quad (\approx 200 \text{ MeV}) \]

and then in the preceding expressions the product \( \kappa p_{/mA} \) may be replaced by \( 10^3 \varepsilon_f \quad (\approx 0.2) \).

3. Balance of Reactivity

Extensive studies on the ADS behavior under incidental conditions are presently made for verifying their claimed advantage, under the safety point of view, with respect to the critical reactors. An extensive analysis of these systems was made in the late 80's at ANL (Wade, 1986), specifically with respect to the Integrated Fast Reactor (IFR). A synthetic, quite effective deterministic method was used based on a "balance of reactivity" approach. We can rewrite a similar methodology in relation to the ADS, making rather use of a "balance of power" approach. This would allow to estimate the behavior of the ADS systems at abnormal conditions. The balance of reactivity approach consists in writing a quasi-static balance of reactivity (\( \rho \)):

\[ \rho = (P - 1)A + (P/F - 1)B + \delta T_{\text{in}} C + \delta \rho_{\text{ext}} = 0 \]
(3.1)

where

\( P \) and \( F \) are power and coolant flow (normalized to unity at operating conditions),

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\(^3\) The number of spallation neutrons per incident proton increases with the energy \( E_p \) (see note 1), and, then, increasing it, the proton current correspondingly decreases for producing the same neutron source intensity.
δT_in is the change from normal coolant inlet temperature T_in,
C is the inlet temperature reactivity coefficient,
(A+B) is the reactivity coefficient experienced in going to full power and flow from zero
power isothermal at constant coolant inlet temperature,
B is the power/flow reactivity coefficient,
δρ_ext is an external reactivity insertion.

In Eq. (3.1) it is assumed that convergence (criticality) has been reached asymptotically. There are circumstances in
which this is not physically possible, as in presence of a scram intervention, implying a strong negative reactivity
insertion (in this case δρ_ext = -|Δρ scram|). This occurrence may be identified, since in these cases the resulting values
δT_in or P/F, in LOHS and LOF events, respectively, with scram intervention, loose physical sense (being negative).

In the following, we consider the same problem starting, rather than from a balance of criticality approach (suitable
for critical systems), from a balance of power one (seemingly, more suitable for subcritical ones).
To note that the formulation proposed shows to be quite general, being applicable to subcritical, as well as to critical
systems.

4. Balance of Power

Whereas in a critical reactor the equilibrium condition after a (limited) incidental transient corresponds to a new
criticality state, in an ADS system in a similar circumstance, due to the external source presence, and assuming that
the incidental transient does not lead to criticality (an intrinsic prerequisite for this system safety), any subcritical
steady state condition may be generally achieved. The following equations are then proposed, at equilibrium,

\[
\rho = (P-1)A + (P/F - 1)B + \delta T_in C + \delta \rho_{ext}
\]

\[
P = \alpha \frac{s_n + \delta s_n}{1 - K_{eff} (1 + \rho/K_{eff})},
\]

K_eff being the multiplication coefficient of the subcritical system before the accidental event, \( \rho \) the reactivity
associated with the deviation of the multiplication coefficient from \( K_{eff} \), and \( \delta s_n \) a change of the external neutron
source \( s_n \), induced by a change \( \delta i \) of the accelerator current \( i \).

With the power unit definition (at nominal conditions)

\[
P_o = \alpha \frac{s_n}{1 - K_{eff}} = 1
\]

we have

\[
\alpha = \frac{1 - K_{eff}}{s_n}
\]

Eq.(4.2) then may be written
\[
P = \frac{1 - K_{\text{eff}}}{\rho + \delta s_n} \cdot \frac{s_n + \delta s_n}{1 - K_{\text{eff}} (1 + \rho K_{\text{eff}})}.
\]

Substituting expression (4.1) for reactivity \(\rho\) and indicating by \(\overline{\rho}\) the subcriticality \((1-K_{\text{eff}})\), we obtain

\[
P[\overline{\rho} - (P - 1)A - (P/F - 1)B - \delta T_{\text{in}} C - \delta \rho_{\text{ext}}] - \overline{\rho}(1 + \frac{\delta i}{i}) = 0
\]

where we have set \(\frac{\delta i}{i}\), i.e., the fractional change of the accelerator current, in place of \(\frac{\delta s_n}{s_n}\).

This equation is quite general. In fact, if the external source term \(\overline{\rho}(1 + \frac{\delta i}{i})\) vanishes (since for a critical system it is \(K_{\text{eff}}=1\)), we obtain again Eq.(3.1).

Eq.(4.6) may be solved with respect to \(P\). We easily obtain

\[
P = \frac{\overline{\rho} + (A + B) - \delta T_{\text{in}} C - \delta \rho_{\text{ext}} - \sqrt{[\overline{\rho} + (A + B) - \delta T_{\text{in}} C - \delta \rho_{\text{ext}}]^2 - 4\overline{\rho}(A + \frac{B}{F} - 1 + \delta i)}}{2(A + \frac{B}{F})}
\]

An important quantity to be analyzed is the coolant output temperature \(T_{\text{out}}\). If by \(\Delta T_c\) we denote the coolant temperature rise at nominal full power/flow ratio, the coolant outlet temperature change \(\delta T_{\text{out}}\) is defined by the expression

\[
\delta T_{\text{out}} = \delta T_{\text{in}} + (\frac{P}{F} - 1)\Delta T_c.
\]

### 4.1 Loss Of Heat Sink (LOHS)

In this case the inlet temperature \(T_{\text{in}}\) increases while the coolant flow remains constant. A dynamic study should be done to analyze the heat balance evolution. However, some qualitative considerations can be made.

Consider two possibilities:

- **The current shuts off.**
  The coolant flow remains unchanged, while \(P \to 0\). It is found, from Eq.(4.6)\(^*\), having set \(\delta i = -i\) (and \(\delta \rho_{\text{ext}} = 0\)),

\[
\delta T_{\text{in}} = \frac{A + B}{C} + \frac{\overline{\rho}}{C}
\]

\(^*\) In cases like this, in which the external source term \(\overline{\rho}/(1 + \delta i/\overline{i})\) vanishes, in Eq. (4.6) the power \(P\) multiplying the terms in square parenthesis is dropped. In fact these terms correspond to the asymptotic overall reactivity, which, in case of equilibrium (criticality) convergence, should vanish. As said previously, if the criticality is not attainable, the \(\delta T_{\text{in}}\) value would result negative, i.e., out of physical sense.
which corresponds to the analogous expression for the IFR system, with at right side of the (negative) $\frac{\bar{\rho}}{C}$ term in place of $\frac{|\Delta \rho_{\text{scram}}|}{C}$, relevant to the (negative) reactivity insertion with the scram intervention.

Since, as power decreases, the outlet temperature $T_{\text{out}}$ collapses into $T_{\text{in}}$, we can also write, recalling Eq.(4.8),

$$\delta T_{\text{out}} = \delta T_{\text{in}} - \Delta T_c = \left[\frac{(A+B) + \bar{\rho}}{C \Delta T_c} - 1\right] \Delta T_c$$

(4.10)

So, for a subcritical system, there is a reduction of $\delta T_{\text{out}}$ with the decreasing of $(A+B)/C$ (usually, a prevailing term) and the increasing (in absolute value) of the negative term $\bar{\rho}/C$.

- The current fails to be shut-off (LOHSWS, Loss of Heat Sink Without current Shut-off)

From Eq.(4.7) it can be shown that in this case the power is sustained down to a lower limit proportional to $\bar{\rho}$. The integrated energy, if not adequately absorbed by the system heat capacity, may lead to unacceptable temperature levels. It is then essential in this case that some intrinsic device is introduced which stops the insertion of external neutrons. In general, it can be said that, in relation to this event, a relatively small value of $\bar{\rho}$, i.e., a relatively small subcriticality level (and, correspondingly, a relatively small neutron source $s_n$), is desirable to limit the intensity of the asymptotic power and, consequently, the value of the outlet temperature (before a corrective intervention takes place).

4.2. Current-related Transient of Over-Power (TOC)

For an ADS a TOC event (analogous to the TOP event of the IFR) may be defined as a current increase $\Delta i_{\text{TOC}}$ at nominal operation level. This change may correspond, for instance, to the reserve of current for compensating reactivity loss with burn-up. The coolant flow $F$ remains unchanged.

Short term ($T_{\text{in}}$ unchanged)

Eq.(4.7) in his case becomes

$$P = \frac{\bar{\rho} + (A+B) - \sqrt{[\bar{\rho} + (A+B)]^2 - 4\bar{\rho}(A+B)(1 + \frac{\Delta i_{\text{TOC}}}{i})}}{2(A+B)}$$

(4.11)

Assuming that $\Delta i_{\text{TOC}}/i$ is a small quantity with respect to unity, we obtain

$$P = 1 - \frac{\bar{\rho}}{(A+B) - \bar{\rho}} \frac{\Delta i_{\text{TOC}}}{i}$$

(4.12)

** In Wade's formulation this term, for a critical reactor, would result

$$\delta T_{\text{in}} = \frac{A + |\Delta \rho_{\text{scram}}|}{C},$$

For comments on this, see Section 2.
The quantity
\[-\bar{p} \frac{\Delta i_{\text{TOC}}}{i}\] (4.13)

may be viewed as the reactivity loss $-\Delta \rho_{\text{TOC}}$ ($\Delta \rho_{\text{TOC}}$ being a positive quantity), during reactor operation and life, to be compensated by the current reserve margin to which $\Delta i_{\text{TOC}}$ corresponds. Eq.(4.12) then can also be written
\[P = 1 - \frac{\Delta \rho_{\text{TOC}}}{(A + B) - \bar{p}}, \] (4.14)

which is quite similar to the expression relevant to the IFR for the corresponding TOP event [to which it corresponds exactly if we set $K_{\text{eff}}$ equal to unity, as may be easily verified from Eq.(4.1)].

We can then write, recalling Eq.(4.8),
\[\delta T_{\text{out}} = -R \frac{\Delta \rho_{\text{TOC}}}{(A + B) - \bar{p}} \Delta T_{c}. \] (4.15)

As with the IFR, given a reactivity margin ($\Delta \rho_{\text{TOC}}$) to be accommodated as a current reserve, also for an ADS system a large value (in absolute terms) of the sum $(A+B)$ would then be desirable. The subcriticality condition also significantly helps under this respect.

Long term ($P \to 0$)

$\delta T_{\text{in}}$ gradually increases until an adequate subcriticality is reached. It is found, from Eq.(4.6),
\[\delta T_{\text{out}} (\equiv \delta T_{\text{in}}) = -\bar{p} \frac{\Delta i_{\text{TOC}}}{C} \equiv -\frac{\Delta \rho_{\text{TOC}}}{C}, \] (4.16)

an expression quite similar to that relevant to the TOP event of IFR (as may be easily found from Eq.(4.1), with $\delta \rho_{\text{ext}} = \Delta \rho_{\text{TOP}}$). A large value of $C$ would be in this case desirable.

4.3. Loss of Flow (LOF)

With this event the inlet temperature $T_{\text{in}}$ is assumed not to change while the coolant flow will coast down to natural circulation. A dynamic study should also here be done to analyze the heat balance evolution. However, some qualitative consideration can be made.

We again consider two possibilities:

- The current is shut off.

In this case $P \to 0$. From Eq. (4.6)* we obtain, at long term,

* In this case, in analogy with what said for the LOHS with scram event, if the equilibrium convergence is not attainable, the P/F value would result negative, i.e., out of physical sense.
\[
\frac{P}{F} = 1 + \frac{A}{B} + \frac{\bar{\rho}}{B}
\]  
(4.17)

which looks like the analogous expression for the IFR case with \( \frac{\bar{\rho}}{C} \) term in place of \( \frac{|\Delta \rho_{\text{scram}}|}{C} \), relevant to the (negative) reactivity insertion with the scram intervention.**

Then

\[
\delta T_{\text{out}} = (\frac{A}{B} + \frac{\bar{\rho}}{B}) \Delta T_{c}
\]  
(4.18)

In this case, a small \( \frac{A}{B} \) and a large \( \frac{\bar{\rho}}{B} \) (intended in absolute value since this is a negative quantity) would be desirable. To notice in this case that the extra term \( \frac{\bar{\rho}}{B} \Delta T_{c} \) helps reducing the \( \delta T_{\text{out}} \) value with respect to the IFR case.

At short term, at which dynamic effects make the system depart from equilibrium and which need be taken into proper account, considerations similar to those expressed with respect to the IFR can be made, in particular those relevant to the pump coast-down time (\( \tau \)). With the ADS, however, the situation would also in this circumstance be alleviated by a large \( \frac{\bar{\rho}}{B} \) (absolute) value.

- The current fails to be shut-off (LOFWS, Loss of Flow Without current Shut-off).

In this case the power is sustained down to a lower limit. As with the LOHSWS case, the integrated energy, if not adequately absorbed via natural circulation, may lead to unacceptable temperature levels. It is then essential also for this case that some intrinsic device is introduced which stops the insertion of external neutrons.

For very small coast-down values of the coolant flow \( F_{\text{NC}} \) (of the order of 1% of the nominal flow), from Eq.(4.7) we would obtain

\[
P \sim \sqrt{\frac{F_{\text{NC}}}{B} \bar{\rho}}
\]  
(4.19)

In a real system, \( F_{\text{NC}} \) would be of the order of 10% of the nominal flow. Eq.(4.7), rather than Eq.(4.19), should therefore be used. The dependence on the subcriticality level \( \bar{\rho} \), however, remains. It can then be said that also in relation to a LOFWS event a relatively small \( \bar{\rho} \) value (and, correspondingly, a relatively small neutron source \( s_n \)), and a large value (in absolute terms) of \( B \) are desirable to limit the intensity of the asymptotic power, and, consequently, the outlet temperature (before a corrective intervention takes place).

** In Wade’s formulation this term, for a critical reactor, would result

\[
\frac{P}{F} = 1 + \frac{A + |\Delta \rho_{\text{scram}}|}{B},
\]

For comments on this, see Section 2.
At short range, the problem associated with the pump coast-down time ($\tau$), is aggravated for an ADS with respect to an IFR by the presence of the persistent external source which may be viewed as an amplification of the delayed neutron holdback problem.

### 4.4. CIT (Chilled Inlet Temperature)

A chilled inlet temperature, or overcooling event (the inverse of a LOHS), inducing a negative change $\delta T_{\text{in}}$ of the coolant inlet temperature, may occur if a steam-line rupture overcools the secondary coolant which in turn overcools the primary core inlet temperature. At constant pump flow the resulting reactivity increase is compensated by a power increase with resultant core temperature rise increase. From Eq.(4.7), since $C\delta T_{\text{in}}$ is a small (positive) quantity with respect to unity, we may write, assuming the accelerator current intensity maintains constant,

$$P = 1 - \frac{C\delta T_{\text{in}}}{(A + B) - \bar{\rho}}$$

(4.20)

and then, recalling Eq.(4.8),

$$\delta T_{\text{out}} = \left(1 - \frac{C\Delta T_{\text{c}}}{(A + B) - \bar{\rho}}\right)\delta T_{\text{in}}$$

(4.21)

which compares with the similar expression for the IFR system. In our case the core outlet temperature results then reduced by a large power coefficient $(A+B)$, a small inlet temperature coefficient $C$ (both in absolute terms) and, as expected, by a relatively large $\bar{\rho}$ value.

### 4.5. IOR (Insertion of Reactivity)

The asymptotic power following an accidental reactivity insertion $\delta \rho_{\text{ext}}$, during normal operation, is likewise obtained from Eq.(4.7). Since $\delta \rho_{\text{ext}}$ may be assumed small with respect to unity, we may write, assuming that the accelerator current intensity is maintained constant,

$$P = 1 - \frac{\delta \rho_{\text{ext}}}{(A + B) - \bar{\rho}}$$

(4.22)

which is quite similar to the expression relevant to the CIT event, with $\delta \rho_{\text{ext}}$ in place of $C\delta T_{\text{in}}$.

Then

$$\delta T_{\text{out}} = -\frac{\delta \rho_{\text{ext}}\Delta T_{\text{c}}}{(A + B) - \bar{\rho}},$$

(4.23)

Also for this case, then, the core outlet temperature would result reduced by a large power coefficient $(A+B)$ (in absolute terms) and a relatively large $\bar{\rho}$ value.

### 5. General Conclusions

For an ADS system, the following general conclusions can be drawn:
1 - A large negative power coefficient (A+B) would be required for reducing TOC (at short term), CIT and IOR accidents, whereas, inversely, small ones would be needed for limiting the consequences of LOHS events with current cut-off. A trade-off between these two contradictory requirements need to be found, if a relatively large value for the (absolute) value of C is not available.

2 - A small current reserve is desirable (so that \( \frac{\Delta i_{\text{TOC}}}{i} \) is small), for to reducing TOC accidents (at small and long terms). This may be achieved (in a system assumed without reactor life control elements) by compensating the burn-up criticality swing by an adequate internal conversion ratio and burnable neutron poisoning.

3 - A small A/B value, to reduce the consequences of a LOF accident with current cut-off. Since for other effects there are contradictory requirements on these coefficients, some trade-off between coefficients A and B requirements in this case need also to be found.

4 - Some intrinsic safety mechanism should be introduced into the system, to stop the current beam and prevent LOHSWS and LOFWS events.

In Appendix A, the results are shown of a study [Gandini, et al., 1999] in which the above approach has been adopted for some quantitative consideration relevant to ADS systems safety. A comparison is also illustrated between the Russian fast critical reactor BREST (Orlov and Slessarev, 1988, and Adamov, 1994) and an ADS with similar characteristics, in order to evidence the respective peculiarities at accidental conditions.

5. The HGPT Method

We shall describe now heuristical-based methods apt for the analysis of source-driven, subcritical systems, at static and time-dependent conditions. A sensitivity methodology will be illustrated, based on the so called Heuristic Generalized Perturbation Theory (HGPT) method, by which relevant physical quantities, functionals of the neutron density, may be analyzed. To relate this methodology it with previous approaches, a short description is first made. These methods have been widely used in the past in the domain of critical reactor physics and played an important role in the analysis of these systems. The extension of their use to the analysis of ADS reactors seems appropriate.

5.1. Background

Since the beginning of nuclear reactor physics studies, perturbation theory has played an important role. As well known, it was first proposed in 1945 by Wigner to study fundamental quantities such as the reactivity worths of different materials in the reactor core. It is also well known that this first formulation, today widely used by reactor analysts, makes a consistent use of the adjoint flux concept. The advantage of using perturbation theory lies in the fact that instead of making a new, often lengthy direct calculation of the eigenvalue (and then of the real flux) for every perturbed system configuration, a simple integration operation is required in terms of unperturbed quantities.

It is interesting that as early as 1948 Soodak associated to the adjoint flux the concept of importance, viewing it as proportional to the contribution of a neutron, inserted in a given point of a critical system, to the asymptotic power. Along with the introduction of the concept of importance and, parallel to it, along with the development of calculation methods and machines, from the early 60' a flourishing of perturbation methods, at first in the linear domain and then in the nonlinear one, have been proposed for analysis of reactor core physics, shielding,
thermohydraulics, as well as other fields. The perturbation formulations proposed by various authors may be subdivided into three main categories, according to the approach followed in their derivation:

1. The heuristic approach, making exclusive use of importance conservation concepts, adopted first by Usachev (1963) and then extensively developed by Gandini (1967-1987). It will be referred to, in the following, as heuristic generalized perturbation theory (HGPT) method.
3. The differential method, proposed by Oblow (1976) and extensively developed by Cacuci (1980), based on a formal differentiation of the response considered.

Each of the above methods has its own merit, although all of them can be shown equivalent to each other (Greenspan, 1975).

Here we shall discuss the potential applications of the HGPT methodology to the analysis of subcritical systems. A first indication of its potential use with respect to neutron kinetic analysis of critical and noncritical systems (with an external source) and to the possibility of analyzing integral experiments in reactor facilities at subcritical conditions was suggested in 1969 (Gandini). In particular, the neutron and precursor importances associated with a given response was considered. In subsequent articles (Gandini, 1976, 1981), the use of HGPT methods for time-dependent problems was again discussed. In particular, the coupled neutron, precursor and multi-channel temperature field, generally in presence of external neutron and enthalpy sources, was suggested for application of the HGPT methodology in dynamic studies.

Considering the increasing attention being given to the subcritical, accelerator driven systems (ADS), the application of the HGPT methodology for their cycle life analysis was proposed in 1997 (Gandini, 1997) basing on a former procedure (Gandini 1987, 1988) developed for critical reactors. Here, we shall shortly review these works. In particular, the role will be discussed of the importance function associated with the power control, and the definition of the concept of "generalized reactivity", merging into the standard concept of reactivity with the system approaching criticality. Basing on these results, a formulation is finally described of a point kinetic equation, with physically significant coefficients, similar to that presented by Usachev (1955) using the standard adjoint flux as weighting function and basing on a previous work by Hurwitz (1949).

5.2. Theory

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle, the temperature of the outlet coolant. The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones. For all these fields the equation governing the importance function was obtained directly by imposing that on average the contribution to the chosen response from a particle [a neutron, or a nuclide, or an energy carrier] introduced at a given time in a given phase space point of the system is conserved through time ("importance conservation principle"). Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.
Consider a linear particle field density represented by vector $f$ (e.g., the multigroup neutron density field) and a response $Q$ of the type

$$Q = \int_{t_0}^{t_F} \langle s^+, f \rangle dt \equiv \ll s^+, f \gg , \quad (5.1)$$

where $s^+$ is an assigned vector function and where $\langle \rangle$ indicate integration over the phase space. Weighting all the particles inserted into the system, let's assume a source $s$, with the corresponding importance ($f^*$) will obviously give the response itself, i.e.,

$$\ll f^*, s \gg = Q = \ll s^+, f \gg , \quad (5.2)$$

which represents an important reciprocity relationship.

From the first derivations mentioned above the rules for determining the equation governing the importance function $f^*$ were learned (see in Appendix B the derivation of this importance for the case relevant to the neutron field). They imply, in relation to the equation governing $f^*$:

- change of sign of the odd derivatives,
- transposing matrix elements,
- reversing the order of operators,
- substitution of the source $s$ with $s^+$.

The first three rules will be generally called "operator reversal" rules.

The HGPT method was then extended to any field governed by linear operators for which the rules for their reversal were known. In particular, it was extended to the derivative fields, obtained from expanding to first order, around a given starting solution, a number of important nonlinear equations, as those governing:

- the coupled neutron/nuclide field, relevant to core evolution and control problems,
- the temperature field, relevant to thermohydraulics.

5.3. General formulation.

Consider a generic physical model defined by a number of parameters $p_j$ ($j=1,2,...,J$) and described by an $N$-component vector field $f$ obeying a generally non-linear equation

$$m(f|p) = 0 . \quad (5.3)$$

Vector $f(q,t)$ generally depends on the phase space coordinates $q$ and time $t$. Vector $p$ represents the set of independent parameters $p_j$ ($j=1,2,...$) fully describing the system and entering into Eq.(5.3). Their value generally

---

* Expression (5.1) is also representative of more general responses, of the type $Q = \ll L(f) \gg$, $L$ being a given function of $f$. In fact, if we extend $f$ to the field $\tilde{f} = \begin{bmatrix} f \\ y \end{bmatrix}$, where $y=L(f)$, $Q$ reduces to the form of Eq.(1), i.e., $Q = \ll s^+ \tilde{f} \gg$, having set $s^+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. 

---
determines physical constants, initial conditions, source terms, etc. Equation (5.3) can be viewed as an equation comprising linear, as well as nonlinear, operators and is assumed to be derivable with respect to parameters $p_j$ and (in the Frechet sense) component functions $f_n$ ($n=1,2,...,N$).

Consider now a response of interest, or functional $Q$ given by Eq.(5.1). In the following, we shall look for an expression giving perturbatively the change $\delta Q$ of the response $Q$ in terms of perturbations $\delta p_j$ of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to each parameter $p_j$ will be obtained.

Expanding equation (5.3) around a reference solution gives, setting $f_j = \frac{df}{dp_j}$, we obtain

$$\sum_{j=1}^{J} \delta p_j (Hf_j + m_j) + O_2 = 0 ,$$

where $O_2$ is a second, or higher order term, and where $m_j = \frac{\partial m}{\partial p_j}$.

Operator $H$ is given by the expression

$$H = \begin{vmatrix}
\frac{\partial \bar{m}_1}{\partial f_1} & \frac{\partial \bar{m}_1}{\partial f_2} & \cdots & \frac{\partial \bar{m}_1}{\partial f_N} \\
\frac{\partial \bar{m}_2}{\partial f_1} & \frac{\partial \bar{m}_2}{\partial f_2} & \cdots & \frac{\partial \bar{m}_2}{\partial f_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \bar{m}_N}{\partial f_1} & \frac{\partial \bar{m}_N}{\partial f_2} & \cdots & \frac{\partial \bar{m}_N}{\partial f_N}
\end{vmatrix} \tag{5.5}$$

where by $\frac{\partial}{\partial f_n}$ we have indicated a Frechet derivative (Listernick and Sobolev, 1972).

Since parameters $p_j$, and then their changes $\delta p_j$, have been assumed independent from each other, it must follow

$$Hf_{j} + m_j = 0 , \tag{5.6}$$

which represents the (linear) equation governing the derivative functions $f_j$. The source term $m_j$ is here intended to account also, via appropriate delta functions, for the initial and, if the case, boundary conditions.

Consider now functional

$$Q_j = \langle \langle \mathbf{h^*}, f_j \rangle \rangle \tag{5.7}$$

Introducing the importance ($f^*$) associated with field $f_j$, if we use it as weight of the source term $m_j$, and integrate space- and time-wise, according to the source reciprocity relationship, Eq.(5.2), the resulting quantity will be equivalent to functional $Q_j$, i.e.,
\[ Q_j = \langle \langle f^*, m_j \rangle \rangle, \quad (5.8) \]

where the importance \( f^* \) obeys the (index-independent) equation

\[ H^* f^* + h^* = 0, \quad (5.9) \]

\( H^* \) being obtained by reversing operator \( H \). As said above, this implies transposing matrix elements, changing sign of the odd derivatives, inverting the order of operators.

We can easily see that the sensitivities \( s_j \) \((j=1,2,\ldots,J)\) of system parameters can be written

\[ s_j = \frac{dQ}{dp_j} = \langle \langle \frac{\partial h^*}{\partial p_j} f \rangle \rangle + \langle \langle f^*, \frac{\partial m}{\partial p_j} \rangle \rangle, \quad (5.10) \]

where the first term at the right-hand side represents the so called, easy to calculate, direct term.

The overall change \( \delta Q \) due to perturbations \( \delta p_j \) \((j=1,2,\ldots,J)\) of system parameters can be written, at first order,

\[ \delta Q = \sum_{j=1}^{J} \delta p_j \left[ \langle \langle \frac{\partial h^*}{\partial p_j} f \rangle \rangle + \langle \langle f^*, \frac{\partial m}{\partial p_j} \rangle \rangle \right]. \quad (5.11) \]

It may occur, in certain circumstances, that one or more components (e.g., \( f_2 \)) of the vector field \( f \) do not depend on a given space-time coordinate (e.g., \( x \)). Consistently with viewing components of \( f \) as (pseudo)-density functions, and without alteration of the problem specifications and results, this, or these variables may be interpreted as averaged, or integral quantities and then replaced by the proper averaging, or integral operator [e.g., \( \frac{\langle \langle \rangle \rangle}{V_x} \), or \( \langle \langle \rangle \rangle \)] applied to the corresponding extended variable [so replacing, to exemplify, \( f_2 \) with \( \frac{\langle \langle f_2(x) \rangle \rangle}{V_x} \), or, simply, \( \langle \langle f_2(x) \rangle \rangle \)]. These extended variables will then be assumed to depend also on this coordinate, although only their average, or integrated values with respect to it are of interest and no further specification for them is required. This rule is referred to as "coordinate dependence complementation". Its use is required in order that a correct operation reversal is made to obtain the operator governing the importance function. In particular, the above rule applies also in those cases in which the response \( Q \), rather than by Eq.(5.1), is given by an expression

\[ Q = \int_{t_0}^{t_f} L(\hat{f} | p) dt, \quad (5.12) \]

\( L(\hat{f} | p) \) being given in terms of integral quantities [for instance, a ratio of the type \( \frac{\langle \langle w_1^+, \hat{f} \rangle \rangle}{\langle \langle w_2^+, \hat{f} \rangle \rangle} \)]. Consistently with the above complementation rule, we shall generally consider field \( f \) defined as \( \langle \langle \hat{f} \rangle \rangle \), with variable \( \hat{y} \) such that \( \langle \langle \hat{y} \rangle \rangle = L(\hat{f} | p) \). The standard expression of the response given by Eq.(5.1), will then apply. The governing Eq.(5.3) will correspondingly become
6. Source Driven Systems

The HGPT methodology was adopted for the sensitivity analysis of the nuclide/neutron core cycle evolution of critical, and may be as well used for the analysis of source driven, subcritical systems. We shall focus here our attention to the methodology applied to their core evolution and kinetic behaviour.

6.1. Core evolution

One of the advantages often claimed for the subcritical source driven power systems is associated to the fact that the power level may be basically controlled by the source strength (via the regulation of the accelerator current). So, no control, or regulating elements would be necessary, if a sufficient breeding is available (and/or an appropriate core burnable poison distribution is provided at the beginning of cycle) in the core for compensating the reactivity loss during burnup. Instead, to the neutron and fuel nuclide densities, represented by vectors $\mathbf{n}(r,t)$ and $\mathbf{c}(r,t)$, respectively, and defined in the reactor cycle interval $(t_o,t_F)$, a specified intensive source control variable, $\rho(t)$, is associated so that the assigned, overall power history $W(t)$ is maintained, as shown in the equations

$$m_{(n)}(\mathbf{n},\mathbf{c},\rho | p) = -\frac{\partial}{\partial t}\mathbf{n} + B\mathbf{n} + \rho s_n = 0$$

$$m_{(c)}(\mathbf{n},\mathbf{c} | p) = -\frac{\partial}{\partial t}\mathbf{c} + E\mathbf{c} + s_c = 0$$

$$m_{(\rho)}(\mathbf{n},\mathbf{c} | p) = <c,S\mathbf{n}> - W = 0$$

where $B$ is the neutron diffusion, or transport, matrix operator (depending on $\mathbf{c}$ and $\rho$), $E$ the nuclide evolution matrix (depending on $\mathbf{n}$), $s_n$ and $s_c$ are given source terms\(^*\), while

$$S = \gamma \begin{bmatrix} \sigma_{f,1}^j & \ldots & \sigma_{f,G}^j \\ \ldots & \ldots & \ldots \\ \sigma_{f,1}^j & \ldots & \sigma_{f,G}^j \end{bmatrix} V,$$

$\gamma$ being the amount of energy per fission, and $\sigma_{f,g}^j$ the microscopic $g$'th group fission cross-section of the $j$'th heavy isotope. $V$ is the diagonal neutron velocity matrix. Quantities $\gamma$, $V$, $W$ and $\sigma_{f,g}^j$ may be considered generally represented by (or function of) system parameters $p_k$. Source terms $s_n$ and $s_c$ are also parameter dependent. If we introduce the field

\(^*\) $s_n$ is generally assumed zero during burnup, except a delta-like source at $t_o$ to represent initial conditions (usually considered at steady state), whereas $s_c$ is generally given by a sum of delta functions defined at $t_o$ and at given times to account for fuel feed and shuffling operations.
the system of Eqs. (6.1), (6.2) and (6.3) may be represented in the compact symbolic form, Eq. (5.3), and the HGPT methodology described above applied.

Since we generally consider systems at quasi-static, i.e., stationary conditions, the time derivative at second member of Eq. (6.1) may be neglected in the course of the integration process.

Any response, functional of variables \( n, c, \) and \( \rho \), could be considered for analysis. We think instructive to limit here consideration to the response defined by the expression

\[
Q = \rho(t_F) \equiv \int_{t_0}^{t_F} \delta(t-t_F)\rho(t)dt
\]

which corresponds to the relative source strength required at \( t_F \) to assure the power level imposed. We may assume that, at unperturbed conditions, \( \rho(t) = 1 \) in the interval \( (t_0, t_F) \). If some system parameter (for instance, the initial enrichment, or some other material density) is altered, as in an optimization search analysis, it may be of interest to evaluate the corresponding change of \( \rho \) at the end of cycle, to make sure that given upper limit specifications of the source strength are non exceeded.

Along with the HGPT methodology, the equations for the corresponding importance functions result

\[
- \frac{\partial n}{\partial t} = B^* n^* + \Omega_c^* c^* + S^T c \rho^*
\]

\[
- \frac{\partial c}{\partial t} = E^* n^* + \Omega_n^* n^* + S n \rho^*
\]

\[
< n^*, s_n > + \delta(t-t_F) = 0
\]

\( \Omega_c^* \) and \( \Omega_n^* \) being operators adjoint of \( \Omega_c \) \( = \frac{\partial(Ec)}{\partial n} \) and \( \Omega_n \) \( = \frac{\partial(Bn)}{\partial c} \), respectively.

Eq. (6.7) corresponds to an orthonormal condition for \( n^* \).

In order to determine the 'final' value \( n^*(t_F) \) required for starting the integration of Eq. (6.5), in consideration of the nature of the above governing equations, we shall first write \( n^* \) and \( \rho^* \) in the form*

\* The diverging of \( n^*(r,t) \) at \( t_F \) may be explained on physical grounds recalling the meaning of importance (in this case, the contribution to the given response by a neutron with the same space/time coordinates) and considering that the response here is \( \rho(t_F) \), i.e., the control assumed to maintain the power at a prefixed level. A neutron introduced at \( t_F \) into the system would in fact produce a (delta-like) impulse of control \( \rho \) to balance its effect on the power level. Then, the importance associated to such neutron would be characterized by a similar delta-like behavior. A quite similar reasoning applies in relation to the diverging of importance \( \rho^*(t) \) at \( t_F \), considering that its physical meaning corresponds to the contribution to the response [defined as \( \rho(t_F) \)] due to a unit energy insertion at \( t_F \) or, which is the same, to an overall power pulse \( \delta(t-t_F) \).
\[ n^*_F(t) = n^*_F \delta(t - t_F) + \tilde{n}^*_F(t) \]  
\[ \rho^*_F(t) = \rho^*_F \delta(t - t_F) + \tilde{\rho}^*_F(t) \]  

with \( \tilde{n}^*_F(t) \) and \( \tilde{\rho}^*_F(t) \) being finite functions, vanishing at \( t_F \).

Replacing into Eq. (6.5), integrating in the interval \((t_F - \varepsilon, t_F + \varepsilon)\), and then making \( \varepsilon \to 0 \), we obtain the equation

\[ B^* n^*_F + S^T e(t_F) \rho^*_F = 0 \]  

Let us now define \( \tilde{n}^*_F \) as obeying equation

\[ B^* \tilde{n}^*_F + S^T e(t_F) = 0 \]  

We note that \( \tilde{n}^*_F \) corresponds to the importance relevant to functional \( \langle c(t_F), S_F(t_F) \rangle \), i.e., to the system power \( W \). From the source reciprocity relationship (Section 2), we may write

\[ \langle c(t_F), S_F(t_F) \rangle = \langle \tilde{n}^*_F, s_F \rangle = W . \]  

From constraint, Eq. (6.7), we easily obtain

\[ \rho^*_F = -\frac{1}{\langle \tilde{n}^*_F, s_F \rangle} = -\frac{1}{W} \]  

and then

\[ n^*_F = n^*_F \rho^*_F = -\frac{\tilde{n}^*_F}{W} . \]  

From this 'final' value, a recurrent calculation scheme may be defined starting from \( t_F \) and proceeding backward.

Along the HGPT methodology, the sensitivity coefficient relevant to the \( k \)'th parameter \( p_k \) is found as

\[ \frac{\partial \rho(t_F)}{\partial p_k} = \rho^*_F [\langle \tilde{n}^*_F, \frac{\partial}{\partial p_k} (Bn + s_n) \rangle + \langle \frac{\partial}{\partial p_k} (c, S_n) - W \rangle] \bigg|_{t_F} \]

\[ + \int_{t_0}^{t_F} [\langle \tilde{n}^*_F, \frac{\partial}{\partial p_k} (Bn + s_n) \rangle + \langle c^*, \frac{\partial}{\partial p_k} e \rangle + \tilde{\rho}^*_F \frac{\partial}{\partial p_k} (c, S_n) - W \rangle \] \]

with \( \rho^*_F \) given by Eq. (6.13). The first term at right side accounts for effects on \( \rho(t_F) \) due to parameter changes at \( t_F \), in particular, if \( p_k \equiv W \), it gives the (trivial) result \( \frac{\partial \rho(t_F)}{\partial W} = \frac{1}{W} \). The second, integral term accounts for analogous effects on \( \rho(t_F) \) produced by parameter changes at times \( t < t_F \).
Rather than on the source term, a control on the neutron absorption in the multiplying region could be of interest. In this case, the (intensive) control variable \( \rho \) would represent the average penetration of the control elements, or the average density of the soluble boron in the coolant, and then would enter into the (transport, or diffusion) operator \( B \). The orthonormal condition for the neutron importance \( n^* \) would now be, rather than Eq. (6.7),

\[
< n^* , \frac{\partial B}{\partial \rho} n > + \delta(t-t_F) = 0 .
\]

In this case, the sensitivity coefficient with respect to a given parameter \( p_k \) would always be given by Eq. (6.15), with \( n_F^* \) obeying Eq. (6.11), but with

\[
\rho_F^* = \frac{1}{<\n_F^* (\frac{\partial B}{\partial \rho} n + \frac{\partial \alpha}{\partial \rho} s) >} .
\]

In general, a control strategy, by which an automatic resetting of the imposed overall power is actuated, might imply a control intervention on both the neutron source strength and the absorbing elements within the multiplying region. In this case, \( \rho \) (which remains a unique, intensive control variable) would affect both operator \( B \) and the neutron source [in this latter case, via an appropriate \( \rho - \) and parameter dependent coefficient \( \alpha(\rho | p) \), assumed unity at unperturbed conditions]. The distribution between these two control mechanisms could be described by appropriate parameters (subject to perturbation analysis). The sensitivity coefficient, in this case, with respect to a given parameter \( p_k \) would always be given by Eq. (6.15), with \( n_F^* \) obeying Eq. (6.11), but with

\[
\rho_F^* = \frac{1}{<\n_F^* (\frac{\partial B}{\partial \rho} n + \frac{\partial \alpha}{\partial \rho} s) >} .
\]

6.2. Stationary Case

To study a given subcritical system at stationary conditions (which may be interpreted at the beginning of its cycle life), we may consider the same system above in which the neutron source and the nuclide density are assumed time-independent during an arbitrary time interval \( (t_0,t_B) \). We assume that at \( t_0 \) the neutron density \( (n_0) \), as well as the control \( (\rho_0) \) have already reached stationary conditions. So, also these two quantities are time-independent in the same time interval. Their governing equations can then be written, in case the power level is controlled by the source strength,

\[
Bn_0 + \rho_0 s_{n,0} = 0 \tag{6.19}
\]

\[
< c_{o,Sn_0} > - W_0 = 0 . \tag{6.20}
\]

Also here we shall assume that at unperturbed conditions \( \rho_0 = 1 \).

The same equations derived previously are applicable to this case, with the advertence of replacing \( t_F \) with \( t_B \) and setting the coupling operators \( \Omega_c^* \) and \( \Omega_n^* \) appearing in Eqs. (6.5) and (6.6) equal to zero. The sensitivity
coefficient of the response $\rho(t_B)$ \(=\rho(t)\) \(=\rho_o\), i.e., constant in the whole interval \((t_0, t_B)\) relevant to the \(j\)'th parameter \(p_k\) can then be obtained. Since in this case \(c^*\), as well as \(n^*(r,t)\) and \(\tilde{\rho}^*(r,t)\) vanish, recalling Eq. (6.15), we obtain

$$\frac{\partial \rho_o^*}{\partial p_k} = \rho_o^* \left[ \langle n_o^* \right. \frac{\partial}{\partial p_k} \left( B n_o + s_{n,o} \right) \left. > + \frac{\partial}{\partial p_k} (\langle c_o, S n_o > - W_o) \right]$$

(6.21)

where

$$\rho_o^* = -\frac{1}{W_o}$$

(6.22)

and \(n_o^*\) obeys equation

$$B^* n_o^* + S^T c_o = 0$$

(6.23)

If, rather than via the source strength, the power level reset control is assumed to be regulated via neutron absorption, so that the control \(\rho_o\) would enter into operator \(B\), the sensitivity coefficient would be given always by Eq. (6.21), but with

$$\rho_o^* = -\frac{1}{\langle n_o^*, \frac{\partial B}{\partial \rho} n \rangle}$$

(6.24)

We might as well consider a (fictitious) control mechanism affecting the fission source, rather than the neutron absorption, i.e., we might choose as control a coefficient multiplying the fission matrix \(F\) and, therefore, entering into the Boltzmann, or diffusion, operator \(B (=A+\rho_o F)\). The sensitivity coefficient would be given again by Eq. (6.21), but with

$$\rho_o^* = -\frac{1}{\langle n_o^*, F n_o \rangle}$$

(6.25)

6.3. Reactivity of Subcritical Systems

For resetting the power level, we have considered above different control mechanisms to which the following types of equations governing the neutron density may be associated:

$$B(p) n_o + \rho_o s_{n,o}(p) = 0$$

(source control) \hspace{1cm} (6.26)

$$B(\rho_o)p n_o + s_{n,o}(p) = 0$$

(neutron absorption, or fission control) \hspace{1cm} (6.27)
\[ B(p_o \mid p)n_o + \alpha(p_o \mid p)s_{ho}(p) = 0 \]  
(mixed control) \quad (6.28)

where the control and parameter dependence is indicated. Coefficient \( \alpha \) is given and reflects the mixed strategy chosen. Eqs. (6.26), (6.27) and (6.28) may be generally represented by equation

\[ m_{(n,o)}(n_o, p_o \mid p) = 0. \]  
(6.29)

The sensitivity expression (6.21) may be generalized so that

\[ \frac{d\rho_o}{d\rho_j} = -\frac{< n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_j} > + \frac{\partial}{\partial \rho_j}(< c_o, S_n_o > - W_o)}{< n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o} >}, \]  
(6.30)

with \( n_o^* \) obeying Eq. (6.23).

A corresponding perturbation expression may now be obtained. Assuming that the power \( W_o \) appearing in Eq. (6.30) is not subject to perturbation, we may write:

\[ \delta \rho_o = -\frac{< n_o^*, \delta m_{(n,o)} > + < n_o, \delta (S^T c_o) >}{< n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o} >}, \]  
(6.31)

where \( \delta m_{(n,o)} = \sum_j \delta \rho_j \frac{\partial m_{(n,o)}}{\partial \rho_j} \) and \( \delta (S^T c_o) = \sum_j \delta \rho_j \frac{\partial (S^T c_o)}{\partial \rho_j} \).

As said previously, \( \delta \rho_o \) corresponds to the control change necessary to reestablish the power level existing before the perturbation \( \delta m_{(n,o)} \). We may as well say that the perturbation \( \delta m_{(n,o)} \) [and \( \delta (S^T c_o) \)] would produce a power level change equivalent to that produced by a control change \( \delta K_{\rho} \) given by the equation

\[ \delta K_{\rho} = \frac{< n_o^*, \delta m_{(n,o)} > + < n_o, \delta (S^T c_o) >}{< n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o} >}. \]  
(6.32)

In the case of the (fictitious) control on the neutron fission, setting \( \lambda \) in place of \( \rho \) to distinguish this peculiar case, we may explicitly write

\[ \delta K_{\lambda} = \frac{< n_o^*, \delta Bn_o > + < n_o^*, \delta s_{n,o} > + < n_o, \delta (S^T c_o) >}{< n_o^*, \delta n_o > + < n_o^*, \delta n_o > + < n_o^*, \delta n_o >}. \]  
(6.33)

\* A mixed control strategy may be considered also using Eq. (6.26), or Eq. (6.27). Adopting, for instance, Eq. (6.26), relevant to the neutron source control, part of the power level would be taken care of parametrically (e.g., by properly changing the control rod position, or the soluble boron density). The remaining reset would be taken care of intrinsically, by the \( \rho \)-control chosen.
The first term at the right side closely resembles the reactivity expression for critical systems. So, we shall call a quantity $\delta K$ as given by expression (6.33) a 'generalized reactivity'. The second term may be defined the "source reactivity", whereas the last one a "direct effect". To account for a generic $\rho$-mode control mechanism, we shall extend this definition to $\delta K_\rho$, similarly defined by Eq. (6.32), i.e.,

$$\delta K_\rho = \frac{<n_o^*, \delta Bn_o>}{<n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o}>} + \frac{<n_o^*, \delta s_{n,o}>}{<n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o}>} + \frac{<n_o^*, \delta(S^T e_o)>}{<n_o^*, \frac{\partial m_{(n,o)}}{\partial \rho_o}>}.$$  

(6.34)

and call it generalized $\rho$-mode reactivity.

6.4. Point Kinetics

Let us now consider equations governing the neutron flux $\phi$ ($\equiv Vn$) and precursor $m_i$ ($i=1,2,...,I$) in a multigroup ($G$ groups) neutron energy scheme:

$$V^{-1} \frac{d\phi}{dt} = A\phi + (1-\beta)\chi_P S_f^G \phi + \chi_D u \sum_{i=1}^I \lambda_i m_i + s_n$$  

(6.35)

$$\frac{dm_i}{dt} = \beta_i \Sigma i^T \phi - \lambda_i m_i$$  

(6.36)

where $A$ is the transport, capture and scattering matrix operator, $V$ the diagonal neutron velocity matrix, $u$ is a unit ($G$ component) vector and

$$S_f^X = \left[ \begin{array}{ccccc} \nu \Sigma_{f,1} & \ldots & \nu \Sigma_{f,G} \\ \ldots & \ldots & \ldots \\ \nu \Sigma_{f,1} & \ldots & \nu \Sigma_{f,G} \end{array} \right]_{(X \text{ rows})}, \quad \Sigma^T = \left[ \begin{array}{cc} \Sigma_{f,1} & \ldots & \Sigma_{f,G} \end{array} \right], \quad \chi_z = \text{diag} \left[ \chi_{z,1} \ldots \chi_{z,G} \right]$$

Setting

$$X_D = \left[ \begin{array}{ccc} \chi_{D,1} & \ldots & \chi_{D,1} \\ \ldots & \ldots & \ldots \\ \chi_{D,G} & \ldots & \chi_{D,G} \end{array} \right]_{(GxI)}, \quad \Lambda = \text{diag} \left[ \lambda_1 \ldots \lambda_1 \right], \quad B = \text{diag} \left[ \beta_1 \ldots \beta_1 \right]$$

Eqs. (6.35) and (6.36) may be written

$$V^{-1} \frac{d\phi}{dt} = A\phi + (1-\beta)\chi_P S_f^G \phi + X_D \Lambda m + s_n$$  

(6.37)

$$\frac{dm}{dt} = BS_f^T \phi - \Lambda m$$  

(6.38)

* The first term at right hand side of Eq. (6.33) can be demonstrated to formally approach the standard reactivity expression as the (reference) system considered gets close to criticality conditions (Gandini, 1997).
or, in matrix form,

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} V^{-1} \phi \\ \mathbf{m} \end{bmatrix} &= \begin{bmatrix} A + (1-\beta)\chi_p S_f^G + X_D \Lambda & \phi \\ BS_f^I & -\Lambda \end{bmatrix} \begin{bmatrix} \phi \\ \mathbf{m} \end{bmatrix} + \begin{bmatrix} s_n \\ 0 \end{bmatrix}
\end{align*}
\]

(6.39)

At unperturbed, steady state conditions Eq. (6.39) reduces to:

\[
\begin{align*}
\begin{bmatrix} A_o + (1-\beta)\chi_p S_f^G & X_D \Lambda \\ BS_f^{1,o} & -\Lambda \end{bmatrix} \begin{bmatrix} \phi_i \\ \mathbf{m}_i \end{bmatrix} + \begin{bmatrix} s_{n,i} \\ 0 \end{bmatrix} &= 0
\end{align*}
\]

(6.40)

or

\[
A_o \phi_o + \left[ \chi_p (1-\beta) + \chi_D \beta \right] S_{f,o} \phi_o + s_n = 0
\]

(6.41)

Consider the neutron importance \( n_{s,o}^* \) associated to the source power control, as defined by Eq. (6.14), and the corresponding precursor density \( \mathbf{m}_{s,o}^* \) (Gandini, 1976). These importances are governed by the equation

\[
\begin{align*}
\begin{bmatrix} A_o^* + (1-\beta)S_{f,o}^{G,T} \chi_p & S_{f,o}^{1,T} B \\ \Lambda X_D^T & -\Lambda \end{bmatrix} \begin{bmatrix} \mathbf{n}_{s,o}^* \\ \mathbf{m}_{s,o}^* \end{bmatrix} &+ \begin{bmatrix} \gamma \Sigma_{f,o} \\ 0 \end{bmatrix} = 0
\end{align*}
\]

(6.42)

\( \gamma \) being the number of energy units per fission and \( W_o \) the system power at stationary, unperturbed conditions.

We may also write:

\[
A_o^* \mathbf{n}_{s,o}^* + \nu S_{f,o}^{T} \left[ (1-\beta)\chi_p + \beta \chi_D \right] \mathbf{n}_{s,o}^* + \frac{\gamma \Sigma_{f,o}}{W_o} = 0
\]

(6.43)

Function \( c_i^* \) results, by definition of importance:

\[
\mathbf{m}_{s,i,o}^* \equiv \mathbf{m}_{s,o}^* = \mathbf{u}^T \chi_D \mathbf{n}_{s,o}^*
\]

(6.44)

Rewrite Eq. (6.39) in the form (writing \( S_f \) rather than \( S_f^G \)):

\[
\begin{align*}
V^{-1} \frac{d\phi}{dt} &= (A_o + \delta A) \phi + (1-\beta)\chi_p (S_{f,o} + \delta S_f) \phi + \chi_D \mathbf{u} \sum_{i=1}^{1} \lambda_i m_i + s_n
\end{align*}
\]

(6.45)

\[
\begin{align*}
\frac{dm_i}{dt} &= \beta_i \nu \Sigma_f^T \phi - \lambda_i m_i
\end{align*}
\]

(6.46)

Multiplying Eqs. (6.45) and (6.46) on the left by \( \mathbf{n}_{s,o}^{T} \), and, \( \mathbf{m}_{s,o}^* \), respectively, space-integrating and recalling expression (6.44), we obtain
\[
\frac{d < n_{s,o}^*, V^{-1} \phi >}{dt} = < n_{s,o}^*, [(A_o + \delta A) + [(1-\beta) \chi_p(S_{i,o} + \delta S_i)] \phi > + \sum_{i=1}^{1} \lambda_i \ < m_{s,o}^* m_i > + < n_{s,o}^*, (s_{n,o} + \delta s_n) >
\] (6.47)

\[
\frac{d < m_{s,o}^* m_{i,o} >}{dt} = \beta_i \ < m_{s,o}^* v_{1f}^T \phi > - \lambda_i \ < m_{s,o}^* c_i >
\] (6.48)

Recalling Eq. (6.43) governing the importance function \( n_{s,o}^* \) and the importance reciprocity relationship

\[
\frac{\gamma}{W_o} < \Sigma_{i,o} \phi_o >= < n_{s,o}^*, s_{n,o} > (= 1),
\] (6.49)

adding and subtracting the term \( \beta \ < n_{s,o}^*, \chi_D S_i \phi > \) at the right side of Eq. (6.47), after some manipulations this transforms into

\[
\frac{d < n_{s,o}^*, V^{-1} \phi >}{dt} = < n_{s,o}^*, \left\{ \delta A + [(1-\beta) \chi_p + \beta \chi_D] \delta S_i \right\} \phi > + < n_{s,o}^*, \delta s_n > + \sum_{i=1}^{M} \lambda_i \ < m_{s,o}^* m_{i,o} > - \beta \ < n_{s,o}^*, \chi_D S_i \phi > + 1 - \frac{W}{W_o} + \frac{\gamma}{W_o} < \delta \Sigma_{i,o} \phi >
\] (6.50)

Let us define the source term

\[
< n_{s,o}^*, \chi S_i \phi o >= (1-\beta) < n_{s,o}^*, \chi_p S_i \phi > + \beta < n_{s,o}^*, \chi_D S_i \phi >
\] (6.51)

and assume that

\[
\frac{d < n_{s,o}^*, V^{-1} \phi >}{dt} = \frac{d}{dt} \left[ \frac{< n_{s,o}^*, V^{-1} \phi > - < n_{s,o}^*, \chi S_i \phi >}{< \Sigma_{i,o} \phi >} \right] = \frac{d}{dt} \left[ \frac{< n_{s,o}^*, V^{-1} \phi > - < n_{s,o}^*, \chi S_i \phi >}{< \Sigma_{i,o} \phi >} \right]
\] (6.52)

If we define then the quantities:

\[
P(t) = \frac{W(t)}{W_o} \quad \text{(relative power)}
\] (6.53)

\[
\ell_{\text{eff}} = \frac{< n_{s,o}^*, V^{-1} \phi >}{< n_{s,o}^*, \chi S_i \phi >} \quad \text{(effective prompt neutron lifetime)}
\] (6.54)
$\rho_{\text{gen}} = \frac{\langle n_{s,0}^*, \delta A + (1 - \beta) \chi_p + \beta \chi_D \rangle \delta S_f \phi_o \rangle + \gamma \langle \delta \Sigma_f \phi_o \rangle}{\langle n_{s,0}^*, \chi S_f \phi_o \rangle}$ (generalized reactivity) \hfill (6.55)

$\rho_{\text{source}} = \frac{\langle n_{s,0}^*, \delta s_a \rangle}{\langle n_{s,0}^*, \chi S_f \phi_o \rangle}$ (source reactivity) \hfill (6.56)

$\alpha = \frac{\langle n_{s,0}^*, \chi_D S_f \phi_o \rangle}{\langle n_{s,0}^*, \chi S_f \phi_o \rangle}$ \hfill (6.57)

$\zeta = \frac{1}{\langle n_{s,0}^*, \chi S_f \phi_o \rangle}$ \hfill (6.58)

$\xi_i = \frac{\langle m_{s,0} m_i \rangle}{\langle n_{s,0}^*, \chi S_f \phi_o \rangle}$ \hfill (6.59)

Eqs. (6.50) and (6.48) may then be written in the form

$$\ell_{\text{eff}} \frac{dP}{dt} = (\rho_{\text{gen}} - \alpha \beta) P + \alpha \sum_{i=1}^{l} \lambda_i \xi_i + \zeta (1 - P) + \rho_{\text{source}}$$ \hfill (6.60)

$$\frac{d\xi_i}{dt} = \beta_i P - \lambda_i \xi_i$$ \hfill (6.61)

with $P=P_e=1$ and $\xi_i = \beta_i/\lambda_i$ at steady state conditions. The expression for $\rho_{\text{gen}}$ was discussed in the previous section.

It is interesting also to note that, with the system approaching criticality, quantity $\zeta$ vanishes. Consequently, the third term at the right side of Eq. (6.60) also vanishes (whereas the space distribution of $n_{s,0}^*$ approaches the standard adjoint flux $\phi_o^*$ (Gandini, 1997). In this case, Eqs. (6.60) and (6.61) reduce to the homogeneous, standard form of the point kinetics equations. Searching solutions for functions $P$ and $\xi_i$ of the form $e^{-\omega t}$, we may arrive at the expression

$$\rho = \ell_{\text{eff}} \omega + \alpha \sum_{i=1}^{l} \frac{\omega \beta_i}{\omega + \lambda_i}$$ \hfill (6.62)

with

$$\rho = \frac{\langle \phi_o^*, \delta A + (1 - \beta) \chi_p + \beta \chi_D \rangle \delta S_f \phi_o \rangle}{\langle \phi_o^*, \chi S_f \phi_o \rangle}$$ \hfill (6.63)
and with $\ell_{\text{eff}}$ and $\alpha$ given by Eqs. (6.54) and (6.57) with $n_{s,o}^*$ replaced by $\phi_o^*$. The general solution will be then given by the superimposition of the solutions corresponding to the (M+1) roots $\omega_i$.

Eqs. (6.60) and (6.61) may be considered an extension of the point kinetic equation to subcritical systems. Solving Eq. (6.62), with $\rho_{\text{gen}}$ given by Eq. (6.63) in place of $\rho$, and with $\ell_{\text{eff}}$ and $\alpha$ given by Eqs. (6.54) and (6.56), shall give the (M+1) roots $\omega_i$ relevant the exponential solutions of the homogeneous equation associated with Eqs. (6.60) and (6.61). As well known, the general solution shall be given by the sum of the solution of the equivalent homogeneous equation and a particular one.

Asymptotically, if after the perturbation the system is still subcritical, a new (relative) power level $P_{\text{as}}$ will be reached, given by the expression

$$P_{\text{as}} = \frac{\zeta + \rho_{\text{source}}}{\zeta - \rho_{\text{gen}}} ,$$

(6.64)

which, as expected, increases with $\rho_{\text{source}}$ and $\rho_{\text{gen}}$.

Quantity $\zeta$ plays the role of a measure of the system subcriticality. To show this, consider first the two subcriticality measures so far generally adopted

$$K_{\text{eff}} = \frac{\langle \phi_o^* \chi S_{f,o} \phi_o^* \rangle}{\langle \phi_o^* s_{n,o} \rangle + \langle \phi_o^* \chi S_{f,o} \phi_o^* \rangle} ,$$

(6.65)

$$K_{\text{source}} = \frac{\langle u \chi S_{f,o} \phi_o^* \rangle}{\langle u s_{n,o} \rangle + \langle u \chi S_{f,o} \phi_o^* \rangle} ,$$

(6.66)

with $u$ a unit vector. $K_{\text{eff}}$ is associated with the fundamental mode of the neutron. It has relevance for safety studies implying accidents bringing the system to overcritical conditions. $K_{\text{source}}$ is a multiplication factor implying the actual flux, in a source driven system generally formed by a superposition of eigenfunctions. It does not take into account the importance of fission and source neutrons with respect to the power. So, taking this importance into account, and recalling that $\langle n_{s,o}^* s_{n,o} \rangle >= 1$, me may define the multiplication coefficient

$$K_{\text{sub}} = \frac{\langle n_{s,o}^* \chi S_{f,o} \phi_o^* \rangle}{1 + \langle n_{s,o}^* \chi S_{f,o} \phi_o^* \rangle} .$$

(6.67)

Quantity $\zeta$ then may be written as

$$\zeta = \frac{1 - K_{\text{sub}}}{K_{\text{sub}}} ,$$

(6.68)

and may be clearly taken as a consistent measure of the distance of the system from criticality.
It was shown (Gandini, 1997) that for $K_{sub}$ approaching unity, function $n_{s,o}^*$ diverges, its space shape approaching that of the standard adjoint flux. Correspondingly, $\rho_{gen}$ converges to the standard form of reactivity, Eq. (6.63).

We have seen that the quantity $\rho_{gen}$ plays a role analogous to that of the reactivity in the point kinetics equation for critical systems. We may also verify that this quantity, for the same parameter perturbation, gives a decreasing contribution to the power change with the system subcriticality increasing. This is due to the presence of the source-related term $\zeta(1-P)$ at the right side of Eq.(6.60), where $\zeta$ increases with the subcriticality.

As we have seen, the coefficients appearing in Eqs.(6.60) and (6.61) are all physically meaningful. The generalized reactivity, $\rho_{gen}$, in particular, may be determined by measurement. In fact, as shown in the previous section, it is given by the product of the source-mode generalized reactivity $\rho_{gen,s}$ associated with the source control [cfr. Eq.(6.21)],

$$\rho_{gen,s} = <n_{s,o}^*\{\delta A + (1 - \beta)\chi_P + \beta\chi_D]\delta S_f\} \phi_o + \frac{\gamma}{W_o} <\delta\Sigma_f \phi_o>, \quad (6.69)$$

by the quantity $\zeta$, given by expression (6.58). Since $\rho_{gen,s}$ corresponds to the source strength change necessary to reset the power level after the perturbation, it is clearly a measurable quantity. For what concerns $\zeta \equiv \frac{1 - K_{sub}}{K_{sub}}$, this quantity doesn't seem easily amenable to experimental evaluation. It seems easier to consider the quantity $\left[\frac{1 - K_{eff}}{K_{eff}}\right]$, obtained by substituting $n_{s,o}^*$ with the standard adjoint function $\phi_o^*$, and then measure it via fundamental mode period measurements. $\zeta$ could be then evaluated by multiplying its calculated value by a bias factor, i.e.,

$$\zeta_{exp} = \zeta_{cal} \frac{(1 - K_{eff})^{exp}}{(1 - K_{eff})^{cal}}, \quad (6.70)$$

Of course, a similar procedure could be also followed for determining via a bias factor $\rho_{gen}^{exp}$ starting from the measurement of a standard reactivity value $\rho^{exp}$.

**Illustrative Example**

Let us consider the simple case of one-group, one precursor, infinite system. In this case Eqs. (6.35) and (6.36) become

$$\frac{1}{\nu} \frac{d\phi}{dt} = -\Sigma_c \phi + (1 - \beta)\nu\Sigma_f \phi + \lambda m + s_n \quad (6.71)$$

$$\frac{dm}{dt} = \beta \nu\Sigma_f \phi - \lambda m. \quad (6.72)$$

At unperturbed conditions it is:

$$-\Sigma_{c,o}\phi_o + \nu\Sigma_{f,o}\phi_o + s_{n,o} = 0 \quad (6.73)$$
The importance function \( n_{s,o}^* \) is governed by the equation

\[
-\Sigma_{c,o} n_{s,o}^* + v \Sigma_{f,o} n_{s,o}^* + \frac{1}{W_o} \gamma \Sigma_{f,o} = 0
\]  

with the solution

\[
n_{s,o}^* = \frac{1}{W_o} \frac{\gamma \Sigma_{f,o}}{\Sigma_{c,o} - v \Sigma_{f,o}} \equiv \frac{1}{W_o} \frac{\gamma K_o}{v 1 - K_o}
\]  

As the (reference) system approaches criticality, and then \( s_{n,\infty} \) for the same power, goes to zero, the importance \( n_{s,o}^* \) diverges. If, on the contrary, it become increasingly subcritical, it correspondingly reduces, vanishing with \( \Sigma_{f,o} \) approaching zero. This is expected recalling the meaning of importance.

Consider a perturbation altering the system parameters. The governing equations will result

\[
\frac{dP}{dt} = \ell_{\text{eff}} = (\rho_{\text{gen}} - \beta)P + \lambda \xi + \frac{1}{W_o} \frac{1 - K_o}{K_o} (1 - P) + \rho_{\text{source}}
\]

(6.78)

\[
\frac{d\xi}{dt} = \beta P - \lambda \xi.
\]

(6.79)

If after the perturbation the system is still subcritical, the new power asymptotic level will be

---

An importance function (Gandini, 1987) is strictly associated with a response defined in a given space interval (at a limit, at a given time point). To exemplify, with the above one-group, infinite medium, the importance \( f^* \) relevant to the power defined at an arbitrary time \( t' \) would be governed by the equation:

\[
-\frac{df^*}{dt} = -\Sigma_{c,o} f^* + v \Sigma_{f,o} f^* + \frac{1}{W_o} \gamma \Sigma_{f,o} \delta(t - t')
\]

(a)

Integrating from \(-\infty\) and \( t' \), recalling that for a subcritical, dissipative system \( f^* \) vanishes for \( t\to-\infty \) and at \( t > t' \), and defining the integrated importance

\[
n_{o}^* = \int_{-\infty}^{t'} f^*(t)dt
\]

(b)

we easily obtain Eq. (6.76). It is also clear that for the system approaching criticality (since the introduction of a neutron at an asymptotic negative time increasingly affects the power value at \( t' \)) the value \( n_{o}^* \) given by Eq. (b) diverges.
\[ P_{as} = \frac{1 - K_o + K_o \rho_{source}}{1 - (K_o + K_o \rho_{gen})} \]  

(6.80)

As expected, the condition for remaining at subcriticality condition is that \( \rho_{gen} < \frac{1 - K_o}{K_o} \).

Assuming the values: \( K_{ef}=0.95 \), \( \ell_{eff} =10^{-3} \), \( \lambda=0.3 \), \( \beta=0.007 \), a number of cases have been evaluated relevant to different reactivity insertions as shown in Figg. 1 through 5 (showing P vs. sec).

Fig. 1. \( \rho_{gen} = 0.005 \) (asymptotic value: \( P=1.11 \))

Fig. 2. \( \rho_{gen} = -0.005 \) (asymptotic value \( P=0.91 \))

Fig. 3. \( \rho_{gen} = 0.0526 \) (critical conditions)

Fig. 4. \( \rho_{gen} = 0.07 \) (over prompt critical)

Fig. 5. \( \rho_{source} = -1 \) (source removal)
Appendix A

As an example of application of the power balance approach described above, the Russian lead cooled fast reactor BREST (Orlov and Slessarev, 1988, and Adamov, 1994) and, for a relative comparison, corresponding ADS systems with various degrees of subcriticality have been considered. The reactivity effects and other relevant characteristics are presented in the Table 1.

**Table 1. Effects and coefficients of reactivity for the BREST reactor.**

<table>
<thead>
<tr>
<th>Effect Description</th>
<th>Coefficient Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lead density variation in the reactor $\alpha_{Pb}$</td>
<td>+ 0.19 pcm/°C</td>
</tr>
<tr>
<td>Radial core expansion $\alpha_R$</td>
<td>- 0.67 pcm/°C</td>
</tr>
<tr>
<td>Assembly plate expansion $\alpha_G = 2\alpha_R$ [Wade]</td>
<td>-1.4 pcm/°C</td>
</tr>
<tr>
<td>Axial fuel elements expansion $\alpha_E$</td>
<td>- 0.11 pcm/°C</td>
</tr>
<tr>
<td>Doppler effect at nominal fuel temperature $\alpha_D$</td>
<td>- 0.43 pcm/°C</td>
</tr>
<tr>
<td>Temperature effect of reactivity $\Delta \rho_{TER}$</td>
<td>- 20 pcm</td>
</tr>
<tr>
<td>Power effect of reactivity $\Delta \rho_{PER}$</td>
<td>- 150 pcm</td>
</tr>
<tr>
<td>Neptunium effect of reactivity $\Delta \rho_{Np}$</td>
<td>- 100 pcm</td>
</tr>
<tr>
<td>Change of isotopic composition due to burnup $\Delta \rho_{FBE}$</td>
<td>+ 30 pcm</td>
</tr>
<tr>
<td>Operational reactivity margin $\Delta \rho_{0P}$</td>
<td>+ 40 pcm</td>
</tr>
<tr>
<td>Total reactivity margin $\Delta \rho_{TOC}$</td>
<td>+ 340 pcm</td>
</tr>
</tbody>
</table>

**Coolant (lead) parameters:**

- inlet temperature - 420 °C,
- outlet temperature - 540°C,
- coolant normal heating $\Delta T_C = 120$°C.
- difference between average fuel and average coolant temperature $T_f = 500$°C.

One can use the following expressions to calculate coefficients A, B and C (Wade, 1986):

- $A = (\alpha_D + \alpha_E) T_f = - 0.75 \beta_{eff}$
- $B = (\alpha_D + \alpha_E + \alpha_{Pb} + \alpha_R) \Delta T_C/2 = - 0.17 \beta_{eff}$
- $C = (\alpha_D + \alpha_E + \alpha_{Pb} + \alpha_G) = -0.0049 \beta_{eff}/°C$.

where $\alpha_G$ is the temperature coefficient relevant to the fuel supporting plate.

**TOC event**

Since one of the possible use of ADS is that of transmutating (incinerating) TRU fuel, systems with solid fuels may be assumed to have a significant burnup reactivity swing, due to the limited breeding available in this case. So, in the example considered, the value $\Delta \rho_{TOC} = 2 \beta_{eff}$ has been assumed.
In a TOP event scenario relevant to a critical reactor it is assumed that all rods run out, this introducing a positive reactivity instantly, whereas in a TOC event scenario relevant to an ADS the accelerator produces the maximum proton current instantly, the coolant flow inlet temperature remaining fixed in both cases at short/intermediate state. All this causes a rise of the power and, then, of the outlet temperature. As time goes on, the inlet temperature starts to rise because the plant cannot absorb the amount of heat produced. In the ideal case, the inlet temperature would increase enough to reduce the power back to its initial level. This corresponds to an asymptotic state. Table 2 and 3 present the results relevant to different levels of subcriticality for ADS and for critical reactors with similar parameters for the TOC event at short/medium term and asymptotic terms, respectively, assuming that $F=F_0$. We note, in particular, that the rise of the outlet temperature in the asymptotic case does not depend on the level of subcriticality, if the alteration, rather than in terms of current change, is given in terms of equivalent reactivity. At the beginning of a TOC transient (short/intermediate state), the ADS system considered has an acceptable temperature rise, compared with the corresponding critical. As expected, the outlet temperature is lower for the lowest $K_{eff}$ condition. Considering asymptotic states, one can conclude that all systems (critical reactors or ADS) may be subject to an excessive temperature rise in correspondence with large $\Delta \rho_{TOC}$ values, of the order of $\Delta \rho_{TOC}=\beta_{eff}$ or higher.

Table 2. TOC parameters ($\Delta \rho_{TOC} = 2\beta_{eff}$) at short/intermediate

<table>
<thead>
<tr>
<th>ADS</th>
<th>Critical reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = 1 - \frac{\Delta \rho_{TOC}}{(A + B)} - \bar{\rho}$</td>
<td>$P = 1 - \frac{\Delta \rho_{TOC}}{(A + B)}$</td>
</tr>
<tr>
<td>$\delta T_{out} = -\frac{\Delta \rho_{TOC}}{(A + B)} \Delta T_c$</td>
<td>$\delta T_{out} = -\frac{\Delta \rho_{TOC}}{(A + B)} \Delta T_c$</td>
</tr>
<tr>
<td>$\bar{\rho} = 10\beta_{eff}$</td>
<td>$\bar{\rho} = 5\beta_{eff}$</td>
</tr>
<tr>
<td>$P = 1.18$</td>
<td>$P = 1.32$ ,</td>
</tr>
<tr>
<td>$\delta T_{out} = 20^\circ C$</td>
<td>$\delta T_{out} = 40^\circ C$</td>
</tr>
<tr>
<td>$T_{out} = 560^\circ C$</td>
<td>$T_{out} = 580^\circ C$</td>
</tr>
<tr>
<td>$\bar{\rho} = 5\beta_{eff}$</td>
<td>$P = 3.2$ ,</td>
</tr>
<tr>
<td>$P = 1.60$</td>
<td>$\delta T_{out} = 260^\circ C$</td>
</tr>
<tr>
<td>$\delta T_{out} = 75^\circ C$</td>
<td>$T_{out} = 800^\circ C$</td>
</tr>
<tr>
<td>$T_{out} = 615^\circ C$</td>
<td>$\bar{\rho} = 2\beta_{eff}$</td>
</tr>
</tbody>
</table>
Table 3. Asymptotic parameters for TOC

<table>
<thead>
<tr>
<th>ADS</th>
<th>Critical reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta T_{out} = -\bar{\rho} \frac{\Delta i_{TOC}}{C} = -\frac{\Delta P_{TOC}}{C}$</td>
<td>$\delta T_{out} = -\frac{\Delta P_{TOC}}{C\Delta T_{C}} \Delta T_{C}$</td>
</tr>
<tr>
<td>All values $\bar{\rho}$</td>
<td>$P \approx 1$</td>
</tr>
<tr>
<td>$P = 1$</td>
<td>$\delta T_{out} = 405^\circ C$</td>
</tr>
<tr>
<td>$\delta T_{out} = 405^\circ C$</td>
<td>$T_{out} = 945^\circ C$</td>
</tr>
</tbody>
</table>

LOHS-WS event

If the process of secondary heat exchange is arrested, in a critical reactor the inlet temperature starts increasing. The negative reactivity effect induced by the inlet temperature rise is compensated by the positive one relevant to the power decrease, up near zero-level. Assuming $F = F_{o}$, then $T_{in} \rightarrow T_{out}$. For an ADS system, there is no equilibrium in the outlet coolant temperature. The outlet temperature is increasing constantly because the ADS power cannot approach zero level, notwithstanding significant feed-backs. The power is sustained down to a lower limit proportional to $\bar{\rho}$. This means that the lower the $K_{e}f$ value, and the higher the neutron source has been chosen, the higher will be the rate of the asymptotic outlet temperature increase. This means that core intrinsic characteristics do not allow to achieve a deterministic safety level, in case of failure of the proton beam stop device.

In Table 4 the results relevant to the coolant asymptotic temperatures for LOHS-WS accidents are shown.

Table 4. LOHS-WS asymptotic parameters

<table>
<thead>
<tr>
<th>ADS</th>
<th>Critical reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td>No equilibrium of outlet coolant temperature</td>
<td>$\delta T_{out} = \left( \frac{A + B}{C\Delta T_{C}} - 1 \right)\Delta T_{C}$</td>
</tr>
<tr>
<td>For all $K_{o}$</td>
<td>$\delta T_{in} = 190^\circ C$</td>
</tr>
<tr>
<td>$T_{out} &gt; 1000^\circ C$</td>
<td>$\delta T_{out} = 70^\circ C$</td>
</tr>
<tr>
<td>$W \rightarrow W_{asympt}$</td>
<td>$T_{out} = 610^\circ C$</td>
</tr>
<tr>
<td>$W \rightarrow 0$</td>
<td>$W \rightarrow 0$</td>
</tr>
</tbody>
</table>

LOF-WS event
With this event the inlet temperature $T_{in}$ is assumed not to change while the coolant flow will coast down to natural circulation. The consequent raising power to flow ratio induces an increase of the core average temperature, this in turn inducing a negative reactivity feedback. This negative reactivity is compensated by a positive one induced by the power reduction. Asymptotically, a natural circulation flow $F_{NC}$ will be established. For preliminary quantitative analysis, one can take $F_{NC} = 0.15 F_0$ at nominal core thermal parameters. Table 5 presents the evaluation of power change as well as the outlet lead temperature growth for ADS at different levels of subcriticality and for the corresponding critical reactor. The results show that the asymptotic temperature level for ADS is unacceptable.

CIT-WS event

With this event an inlet temperature decrease of 100°C has been assumed. In Table 6 the results are given relevant to the power change as well as the outlet coolant temperature increase for ADS at different levels of subcriticality and for the corresponding critical reactor. As far as this type of accident is concerned, the ADS and the corresponding critical system have comparable behaviors.

Table 5. LOF-WS asymptotic parameters ($F_{NC} = 0.15F_0, P_0/F_0 = 1$)

| Table 5. LOF-WS asymptotic parameters ($F_{NC} = 0.15F_0, P_0/F_0 = 1$) |
|---|---|
| ADS | Critical reactor |
| Power decreasing with $\bar{\rho}$ [See Eq.(8)] | \[ P = 1 + \frac{B + A \frac{F_{NC}}{F_0}}{B + A \frac{F_{NC}^2}{F_0^2}} \left( \frac{F_{NC}}{F_0} - 1 \right) \Delta T_C \] |
| $\bar{\rho} = 10 \beta_{eff}$ | $P = 0.93$  |
| $\delta T_{out} = 625°C$ | $T_{out} = 1165°C$ |
| $\bar{\rho} = 5 \beta_{eff}$ | $P = 0.87$  |
| $\delta T_{out} = 575°C$ | $T_{out} = 1115°C$ |
| $\bar{\rho} = 2 \beta_{eff}$ | $P = 0.48$  |
| $\delta T_{out} = 505°C$ | $T_{out} = 800°C$ |
Table 6. CIT-WS asymptotic parameters

<table>
<thead>
<tr>
<th>ADS</th>
<th>Critical reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = 1 - \frac{C \delta T_{in}}{(A + B) - \bar{\rho}}$</td>
<td>$P = 1 - \frac{C \delta T_{in}}{(A + B)}$</td>
</tr>
<tr>
<td>$\delta T_{out} = (1 - \frac{C \Delta T_{c}}{(A + B) - \bar{\rho}}) \delta T_{in}$</td>
<td>$\delta T_{out} = (1 - \frac{C \Delta T_{c}}{(A + B)}) \delta T_{in}$</td>
</tr>
</tbody>
</table>

All $K_{eff}$ and $\bar{\rho}$

$P \approx 1.05$

$\delta T_{out} < 10^\circ C$

$T_{out} = 550^\circ C$

$P \approx 1.5$

$\delta T_{out} = 35^\circ C$

$T_{out} = 575^\circ C$

**Appendix B**

Let us consider the generic transport equation, with obvious notation,

$$\frac{dn}{dt} = - \nabla \cdot v n - \Sigma_i v n (r, \Omega, E|t) + \int d\Omega' \int_0^\infty \Sigma_s(E' \rightarrow E, \Omega' \rightarrow \Omega) v n(r, E', \Omega'|t) dE' + \frac{\chi(E)}{4\pi} \int d\Omega' \int_0^\infty \Sigma_s(E') v n(r, E', \Omega'|t) dE' + s_n$$

(B.1)

The boundary conditions are obtained from physical considerations. Assuming that the system is isolated, i.e., comprehending all its neutron sources, and that external boundary surfaces are convex, it will be:

Flux $\phi(r, \Omega, E|t)=0$ for directions of $\Omega$ entering in the medium.

Let us consider now in a the interval $(t_o, t_f)$ a generic functional

$$Q = \int_{t_o}^{t_f} \int_{4\pi} d\Omega \int_0^\infty dE \int_{sist} dr h^+ n(r, E, \Omega) = << h^+ n >>$$

(B.2)

with vector function $h^+$ given. The notation $<< >>$ here means integration over space and time.

For times $t < t_f$, we may write the balance equation governing the importance function. Let us see closer the mechanisms by which a neutron of coordinates $(r, E, \Omega)$ gives and gains importance.
At the beginning it will have an amount of importance which we shall denote as $n^* (r, E, \Omega)$. After a time $\Delta t$ the following events will occur:

a) The neutron has reached point $r' = r + \Omega \Delta s$, where

$$\Delta s = v \Delta t$$

keeping the same velocity. The probability for the neutron of arriving at $r'$ is given by the quantity.

$$\left(1 - \frac{\Delta s}{l_t (E)} \right)$$

where

$$l_t (E) = \frac{1}{\Sigma_t (E)}$$

which corresponds to the mean free path of the neutron without undergoing any type of collision.

b) The neutron undergoes a scattering collision with change of energy and angle, respectively, from $E$ into the interval $dE'$ around $E'$ and from $\Omega$ into the interval $d\Omega'$ around $\Omega'$. This occurs with probability:

$$\frac{\Delta s \Sigma_s (E \rightarrow E', \Omega \rightarrow \Omega')dE'd\Omega'}{l_t} = \Delta s \Sigma_s (E \rightarrow E', \Omega \rightarrow \Omega')dE'd\Omega'$$

which corresponds to the product of the probability that during the interval $\Delta s$ the neutron undergoes a collision with and that the collision is a scattering one.

e) The neutron undergoes a fission collision. In analogy with the scattering, the probability that a fission neutron emerges in the interval energy $dE'$ around $E'$ and within $d\Omega'$ around $\Omega'$ is given by the ratio:

$$\Delta s v \Sigma_f (E) \frac{\chi (E')}{4 \pi} dE' d\Omega'$$

e) During the interval $\Delta s$, the neutron contributes to the response $Q$ equal to

$$h^+ (\bar{r}, E, \Omega, \bar{t}) \Delta t$$

f) The neutron undergoes a parasitic capture. In such case it simply disappears from the system.

To the neutrons so emerged after a time $\Delta t$ we may associate the values of the importance function associated with the coordinates which characterize such neutrons. The events to be accounted for are the first four ones. For the importance conservation principle, the sum of the importances relevant to each possible event must be equal to that of the starting neutron. It will then be, recalling that $\Delta s/\Delta t = v$, 
\[ n^*(\mathbf{r}, E, \Omega, t) = (1 - \Sigma_i \Delta s)n^*(\mathbf{r} + \Delta s \mathbf{E}, \Omega, t + \Delta t) \]
\[ + \Delta s \int_{4\pi} d\Omega \int_0^{\infty} dE' \Sigma_s(E \rightarrow E', \Omega \rightarrow \Omega') n^*(\mathbf{r}, E', \Omega', \bar{t}) \]
\[ + v \Sigma_i(E) \frac{\Delta s}{4\pi} \int_{4\pi} d\Omega' \int_0^{\infty} dE' \chi(E') n^*(\mathbf{r}, E', \Omega', \bar{t}) + h^*(\mathbf{r}, E, \Omega, \bar{t}) \frac{\Delta s}{v} \]  

(B.3)

where \( \mathbf{\bar{r}} \) represents a point in the \((\mathbf{r}, \mathbf{r} + \Omega \Delta s)\).

Adding and subtracting at the first member of equations (14.33) \( n^*(\mathbf{r}, E, \Omega, t + \Delta t) \) and dividing by \( \Delta s \), beside the incremental ratio

\[ \frac{n^*(\mathbf{r} + \Delta s \mathbf{E}, \Omega, t + \Delta t) - n^*(\mathbf{r}, E, \Omega, t + \Delta t)}{\Delta s} \rightarrow \frac{\partial n^*(\mathbf{r}, E, \Omega, t)}{\partial t} \]

at the first member there will be also the ratio

\[ \frac{1}{v} \frac{n^*(\mathbf{r}, E, \Omega, t + \Delta t) - n^*(\mathbf{r}, E, \Omega, t)}{\Delta t} \rightarrow \frac{1}{v} \frac{\partial n^*(\mathbf{r}, E, \Omega, t)}{\partial t} . \]

Making \( \Delta t \rightarrow 0 \), we shall then obtain the equation governing the importance function:

\[ -\frac{dn^*}{dt} = \mathbf{\Omega} \frac{\partial n^*}{\partial \mathbf{r}} - \Sigma_i v n^* + v \int_{4\pi} d\Omega' \int_0^{\infty} dE' \Sigma_s(E \rightarrow E', \Omega \rightarrow \Omega') n^*(\mathbf{r}, E', \Omega') \]
\[ + \frac{v \Sigma_i(E) v}{4\pi} \int_{4\pi} d\Omega' \int_0^{\infty} dE' \chi(E') n^*(\mathbf{r}, E', \Omega') + h^* \]  

(B.4)

As may be easily verified, this equation may be obtained from that relevant to neutron density by changing the sign of the first derivatives, exchanging the arguments \((E' \rightarrow E, \Omega' \rightarrow \Omega)\) with \((E \rightarrow E', \Omega \rightarrow \Omega')\), respectively, and, similarly, for what concerns the fission source, substituting to \(\chi(E) v \Sigma_i(E')\) the product \(\chi(E') v \Sigma_i(E)\). In other terms, we may say that the importance function is symmetrical to the real density, this implying a reversion of the operators. This symmetry is reflected also in relation to the boundary conditions. As well known, the boundary conditions associated with the real density, in case of an isolated system, are:

\[ n(\mathbf{r}, E, \Omega) = 0 \quad \text{for } \mathbf{r} \text{ on the external boundary and } \Omega \text{ directed inside the system (assumed having a convex external surface).} \]

On the contrary, the boundary conditions relevant to the importance function are:

\[ n^*(\mathbf{r}, E, \Omega) = 0 \quad \text{for } \mathbf{r} \text{ on the external boundary and } \Omega \text{ directed outside (assumed having a convex external surface).} \]

This condition is obtained considering that the contribution to the response from a neutron escaping from the system is clearly null.
In general, we may define the general principle of symmetry between the real flux and importance function, according to which all the properties valid for the flux are also valid for the asjoint function, provided that the sense of energy, angular, space and time variations are reversed.

**Diffusion approximation**

Let us consider now the multigroup equation in diffusion theory:

\[
\frac{dn_i}{dt} = v_i D_i \nabla^2 n_i - v_i n_i \Sigma_{i,i} + \sum_{j=1}^{i} v_j n_j \Sigma_{j,j\rightarrow i}^0 + \chi_i \sum_{j=1}^{N} v_j n_j \Sigma_{f,j,i} (B.5)
\]

Basing on the previous arguments, the importance function relevant to the corresponding response expressed in vector form as \( Q = \ll \mathbf{h}^+, \mathbf{n} \gg \), will be:

\[
-\frac{dn_i^*}{dt} = v_i D_i \nabla^2 n_i^* - v_i n_i^* \Sigma_{i,i} + v_i \sum_{j=i}^{N} n_j^* \Sigma_{j,j\rightarrow i}^0 + v_i \chi_i \sum_{j=1}^{N} n_j^* \Sigma_{f,j,i}^0 + h_i^+ (B.6)
\]

In this case, since the laplacian \( \nabla^2 \) corresponds do a double derivation in space, its sign doesn't change with respect to the real case, while in the terms of the sum their indeces \( i,j \) are exchanged.

A significant simplification of the notation is obtained by writing the equations in vector representation by introducing matrix operators. In particular, for equations (B.5) and (B.6), relevant to the real flux and the importance function, respectively, we may define the following operators:

\[
B = A + F \quad (B.7)
\]

\[
B^* = A^* + F^* \quad (B.8)
\]

where

\[
A = \begin{pmatrix}
(D_1 \nabla^2 - \Sigma_{t,1} + \Sigma_{t\rightarrow i}) & 0 & \ldots & 0 \\
\Sigma_{1\rightarrow 2} & (D_2 \nabla^2 - \Sigma_{t,2} + \Sigma_{2\rightarrow 2}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{1\rightarrow N} & \Sigma_{2\rightarrow N} & \ldots & (D_N \nabla^2 - \Sigma_{t,N} + \Sigma_{N\rightarrow N})
\end{pmatrix}
\]

\[
F = \begin{pmatrix}
v \Sigma_{f,1} \chi_1 & v \Sigma_{f,2} \chi_1 & \ldots & v \Sigma_{f,N} \chi_1 \\
v \Sigma_{f,1} \chi_2 & v \Sigma_{f,2} \chi_2 & \ldots & v \Sigma_{f,N} \chi_2 \\
\vdots & \vdots & \ddots & \vdots \\
v \Sigma_{f,1} \chi_N & v \Sigma_{f,2} \chi_N & \ldots & v \Sigma_{f,N} \chi_N
\end{pmatrix}
\]

\[
(B.9)
\]

\[
(B.10)
\]
\[
A^* = \begin{bmatrix}
(D_1 V^2 - \Sigma_{t,1} + \Sigma_{1\rightarrow 2}) & \Sigma_{1\rightarrow 2} & \cdots & \Sigma_{1\rightarrow N} \\
0 & (D_2 V^2 - \Sigma_{t,2} + \Sigma_{2\rightarrow 2}) & \cdots & \Sigma_{2\rightarrow N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (D_N V^2 - \Sigma_{t,N} + \Sigma_{N\rightarrow N})
\end{bmatrix}
\]

(B.11)

\[
F^* = \begin{bmatrix}
\nu \Sigma_{f,1} \chi_1 & \nu \Sigma_{f,1} \chi_2 & \cdots & \nu \Sigma_{f,1} \chi_N \\
\nu \Sigma_{f,2} \chi_1 & \nu \Sigma_{f,2} \chi_2 & \cdots & \nu \Sigma_{f,2} \chi_N \\
\vdots & \vdots & \ddots & \vdots \\
\nu \Sigma_{f,N} \chi_1 & \nu \Sigma_{f,N} \chi_2 & \cdots & \nu \Sigma_{f,N} \chi_N
\end{bmatrix}
\]

(B.12)

and the following vectors

\[
\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}, \quad \mathbf{n}^* = \begin{bmatrix} n_1^* \\ n_2^* \\ \vdots \\ n_N^* \end{bmatrix}
\]

(B.13)

Equations (B.5) e (B.6) may then be written in the compact form

\[
\frac{d\mathbf{n}}{dt} = \mathbf{B} \mathbf{v} \mathbf{n}
\]

(B.14)

\[
-\frac{d\mathbf{n}^*}{dt} = \mathbf{B}^* \mathbf{n}^* + \mathbf{h}^*.
\]

(B.15)

To note that the elements off the diagonal of matrices (B.9) and (B.11) correspond to scattering transfer macroscopic cross-sections, while the elements of (B.10) e (B.12) correspond to fission macroscopic ones, multiplied by the number of secondaries per fission. To note also that for obtaining matrix \(A^*\) from \(A\), matrix \(F^*\) from \(F\), and then matrix \(B^*\) from \(B\), rows and columns are exchanged, which corresponds to exchanging group indices \(i,j\).

In this case the boundary condition for the importance function remains the same as that for the real flux, i.e., it vanishes at the extrapolated length.

Writing the above equations in terms of the neutron flux \(\phi\) (=\(V\mathbf{n}\)), we have

\[
\frac{d\phi}{dt} = \mathbf{B} \phi
\]

(B.16)

\[
-\frac{d\phi^*}{dt} = \mathbf{B}^* \phi^* + \mathbf{h}^*.
\]

(B.17)
To note that equations (B.15) and (B.17), relevant to the importance function, are equivalent.

References

Usachev L.N. (1955) 1st ICPUE-UN, 5, 503