Nuclear Data and Integral Measurements Correlation for Fast Reactors.
Part 1: Statistical Formulation

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INTRODUCTION

This is the first of a series of reports concerning the correlation between nuclear data and integral measurements obtained on fast critical facilities. The present one deals with the statistical foundations of the correlation procedure and is largely based on the general lines described by Linnik in his textbook [1], which are specialized here to the problem of exploiting the information relevant to fast reactor nuclear design. This effort has been partly suggested by a few difficulties, and sometimes inconsistencies, which are present in many works so far published, very presumably due to the difficulties encountered when dealing with a new subject(\*). Besides, a clear and self-consistent (only a rather limited knowledge of statistics and matrix algebra being presupposed) explanation in terms specialized to reactor physics seemed useful also for a wider participation and discussion on this field of research. For this purpose a number of explicatory notes and appendices has been added to the main text, whenever it appeared appropriate for a clearer understanding (and, hopefully, discussion).

(\*) A detailed review and comments on some of these works will be the object of another report.
1. - DERIVATION OF THE CORRELATION PROCEDURE

Let us consider a number J of different integral quantities $Q_j$ for a class of reactors. Among these we can mention reaction rate ratios, reactivity worths, etc. If we knew the true values of the cross sections $\sigma_i$ (in number of 1) and assuming that the theoretical model for representing the neutron diffusion is exact, we could express the quantities $Q_j$ as functions of $\sigma_i$, i.e.

$$Q_j = Q_j(\sigma_1, \sigma_2, \ldots, \sigma_I), \quad (j=1,2,\ldots,J) \quad (1)$$

If we assume a given set of values $(\sigma)_{0,i}$, by hypothesis close enough to the true values $\sigma_i$, we may expand Eqn. (1), disregarding second and higher order terms and obtain:

$$Q_j = Q_j(\sigma_{0,1}, \ldots, \sigma_{0,I}) + \sum_{i=1}^{I} \frac{\partial Q_j}{\partial \sigma_i} \bigg|_{\sigma_0} (\sigma_i - \sigma_{0,i}) \quad (2)$$

$(x)$ Generally different from the experimental set considered later.
The integral quantities \( Q_j \) are assumed different from each other so that Eqns. (2) may be considered linearly independent. The values \( Q_j(\sigma_0,1,\ldots,\sigma_0,I) \) may be set corresponding to the calculated values of the integral quantities \( Q_j \) and we shall call them \( Q_j^{\text{cal}} \), i.e.

\[
Q_j^{\text{cal}} = Q_j(\sigma_0,1,\ldots,\sigma_0,I).
\]

With the notation:

\[
\frac{Q_j^{\text{cal}} - Q_j}{Q_j} = y_j \quad (j=1,\ldots,J) \tag{3}
\]

\[
\frac{\sigma_i - \sigma_0,i}{\sigma_0,i} = y_{J+1} \quad (i=1,\ldots,I) \tag{4}
\]

\[
-\delta_{ji} = s_{j,i} \quad (j=1,\ldots,J) \tag{5}
\]

\[
\left. \frac{1}{Q_j^{\text{cal}}} \frac{\partial Q_j}{\partial \sigma_i} \right|_{\sigma_0} = s_{j,J+i} \quad (i=1,\ldots,I) \tag{6}
\]

\[
s_{j,0} = 0 \quad (j=1,\ldots,J) \tag{7}
\]

where \( \delta_{ji} \) is the Kronecker's symbol, Eqn. (2) may be written:

\[
(\star) \quad \text{These (here null) quantities are included for generality. It may occur that their values are set different from zero in special cases (for instance in tests for systematic error search or when derived experimental quantities are considered).}
\]
\[
\begin{align*}
\sum_{i=1}^{J} a_{j,i} y_i + \sum_{i=1}^{I} a_{j,J+i} y_{J+i} &= s_{j,0} + \sum_{i=1}^{I+J} s_{j,i} y_i = 0. \quad (j=1, \ldots, J)
\end{align*}
\]

Let us suppose then that we wish to include further different relationships between the quantities \(y_i\), depending on some specific criteria (for instance, known ratios among the parameters considered) and corresponding to \(H\) equations of the type (8), not linearly depending from each other, i.e.:

\[
\sum_{i=1}^{I+J} s_{j+h,i} y_i = 0. \quad (h=1, \ldots, H) \quad (9)
\]

Setting:

\[
J + H = q \quad J + I = N
\]

and introducing the vectors

\[
\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{J+I} \end{bmatrix}
\]

\((x)\) Here \(s_{j+h,0}\) is generally supposed different from zero.
\[
\mathbf{s}_0 = \begin{bmatrix}
s_{1,0} \\
\vdots \\
s_{q,0}
\end{bmatrix}
\]  \hspace{1cm} (11)

and the matrix
\[
\mathbf{S}_{q,N} = \begin{bmatrix}
s_{1,1} & s_{1,2} & \cdots & s_{1,N} \\
s_{2,1} & s_{2,2} & \cdots & s_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
s_{q,1} & s_{q,2} & \cdots & s_{q,N}
\end{bmatrix}
\]  \hspace{1cm} (12)

Eqns. (8) and (9) may be written symbolically:
\[
\mathbf{s}_0 + \mathbf{S} \mathbf{y} = 0 .
\]  \hspace{1cm} (13)

With the assumption made of the linear independence of the (linear) equations, we may say that the rank of matrix \( \mathbf{S} \) is equal to the number of the constraints \( q \) (which, consequently, cannot exceed the number of variables \( N \)). The vector \( \mathbf{s}_0 \) and the matrix \( \mathbf{S} \) are known, while \( \mathbf{y} \) represents the vector of the true relative corrections to the quantities \( Q_j^{\text{cal}} \) and \( \sigma_{0,i} \), which we want to estimate. What we actually know are (indirect, or derived) experimental values \( y_k^{\text{ex}} \), obtainable by replacing in Eqns. (3) and (4) \( Q_j \) and \( \sigma_{1} \) with the experimental values \( Q_j^{\text{ex}} \) and \( \sigma_{1}^{\text{ex}} \), respectively. We obtain:
\[
\begin{align*}
\frac{Q_{j}^{\text{ex}} - Q_{j}^{\text{cal}}}{Q_{j}^{\text{cal}}} &= y_{j}^{\text{ex}} \quad (j=1,\ldots,J) \quad (14) \\
\frac{\sigma_{i}^{\text{ex}} - \sigma_{i}^{\text{cal}}}{\sigma_{i}^{\text{cal}}} &= y_{j+i}^{\text{ex}} \quad (i=1,\ldots,I) \quad (15)
\end{align*}
\]

Let us now assume that the measured quantities \(Q_{j}^{\text{ex}}\) and \(\sigma_{i}^{\text{ex}}\) (and therefore \(y_{k}^{\text{ex}}\)) have a normal distribution of errors with a dispersion matrix \(\mathbb{B}_{y}\), the \((i,j)\)-th element of which is given by the expected value

\[
E[(y_{i}^{\text{ex}} - y_{i})(y_{j}^{\text{ex}} - y_{j})]
\]

where \(y_{i}\) represents a true value (in particular, the \((i,i)\)-th element represents the variance of the experimental value \(y_{i}^{\text{ex}}\)). The likelihood function of vector \(\mathbf{y}\) may therefore

\(\text{\textsuperscript{(*)}}\) No correlation is presupposed between the integral data measurements, having dispersion matrix \(\mathbb{B}_{Q}\), and the cross section experimental values, having dispersion matrix \(\mathbb{B}_{r}\). Therefore, the dispersion matrix \(\mathbb{B}_{y}\) is of the form

\[
\mathbb{B}_{y} = \begin{bmatrix} \mathbb{B}_{Q} & 0 \\
0 & \mathbb{B}_{r} \end{bmatrix}.
\]
be written as

\[ L(y|y^{ex}) = \frac{1}{(2\pi)^{\frac{1}{2}}(\det \Sigma_y)^{\frac{1}{2}}} \exp \left( -\frac{1}{2}(y^{ex} - y)^T \Sigma_y^{-1} (y^{ex} - y) \right). \]  

(16)

This function is maximized if we chose an estimator \( \tilde{y} \) of \( y \) such that

\[ (y^{ex} - \tilde{y})^T \Sigma_y^{-1} (y^{ex} - \tilde{y}) = \text{Minimum} \]  

(17)

and the constraints represented by the system:

\( \bullet \) The maximum likelihood function is formally identical with the simultaneous distribution of \( y^{ex} \), with the difference that, while in this distribution \( y^{ex} \) represents a vector of variables and \( y \) a vector of constant parameters, in the likelihood function \( y^{ex} \) represents a vector of fixed values and \( y \) a vector of parameters of which optimal estimators are to be determined. These may be shown to be obtained by searching that vector \( \tilde{y} \) of estimators which maximizes the likelihood function itself, as may be expected on simple intuitive grounds. In fact, since the values \( y^{ex} \) have really occurred, it seems natural that the probability density around these values (which represent all the experimental data available on which to base the estimate) should be high, on the average. In Appendix 3, which is related with the non-normality of the errors, it is shown how the estimators obtained by the maximum likelihood method, in which the normality of errors has been kept, result having optimal properties, i.e. unbiasedness and minimal variance, with respect to estimators obtained with any other method. This result may serve also as an "a posteriori" demonstration of the optimality of the results obtained by the maximum likelihood method itself, in particular when a normal distribution of errors is considered.
\[ b_0 + \int \tilde{y} = 0 \] (18)

are satisfied.

If we set

\[ v = \bar{y} - \bar{y}^{ex} \] (19)
\[ \tilde{v} = \tilde{y} - \bar{y}^{ex} \] (20)
\[ a_0 + \int \tilde{y}^{ex} = m, \] (21)

Eqns. (17) and (18) may be written in terms of the estimators \( \tilde{y} \) of the true corrections \( v \) of the quantities \( \bar{y}^{ex} \):

\[ \tilde{v}^T \Omega^{-1} \tilde{v} = \text{minimum} \] (22)
\[ m + \int \tilde{v} = 0. \] (23)

Vector \( \tilde{v} \) may be evaluated by means of the Lagrange multipliers \( k_1, k_2, \ldots, k_d \). Introducing the vector

\[ k = \begin{bmatrix} k_1 \\ \vdots \\ k_d \end{bmatrix} \]

the Lagrange function takes the form:

\[ \Psi = \tilde{v}^T \Omega^{-1} \tilde{v} - k^T (m + \int \tilde{y}) \] (25)
Differentiation of the first and second term at the right hand side leads to:
\[
d(\mathbf{y}^T \mathbf{\beta}_y^{-1} \mathbf{y}) = 2 \mathbf{y}^T \mathbf{\beta}_y^{-1} d\mathbf{y} \tag{26}(*)
\]
\[
d[\mathbf{k}^T (\mathbf{m} + \mathbf{\beta}_y \mathbf{y})] = \mathbf{k}^T \mathbf{\beta}_y d\mathbf{y} . \tag{27}
\]

Therefore, setting
\[
d\psi = 2(\mathbf{y}^T \mathbf{\beta}_y^{-1} \mathbf{y} - \frac{1}{2} \mathbf{k}^T \mathbf{\beta}_y \mathbf{\beta}_y \mathbf{k}) d\mathbf{y} = 0 \tag{28}
\]
we obtain the condition:
\[
\mathbf{y}^T \mathbf{\beta}_y^{-1} \mathbf{y} - \frac{1}{2} \mathbf{k}^T \mathbf{\beta}_y \mathbf{k} = 0 \tag{29}
\]
which, together with the constraints
\[
\mathbf{m} + \mathbf{\beta}_y \mathbf{y} = 0 \tag{30}
\]
allows to determine the vector \( \mathbf{\hat{y}} \) (and, if requested, the vector of multipliers \( \mathbf{k} \)). From Eqn. (29) we obtain:
\[
\mathbf{\hat{y}} = \frac{1}{2} \mathbf{\beta}_y \mathbf{\beta}_y^T \mathbf{k} \tag{31}
\]
and introducing this equation in Eqn. (30):

\(*)\) This expression is correct since matrix \( \mathbf{\beta}_y^{-1} \) is symmetric.
\[ k = -2 \mathbf{g}^{-1} \mathbf{m} \]  \hspace{1cm} (32)

where

\[ \mathbf{g} = \mathbf{B}_y \mathbf{f}^T. \]  \hspace{1cm} (33)

Finally, from Eqn. (31) we obtain the result:

\[ \tilde{\mathbf{y}} = -\mathbf{B}_y \mathbf{f}^T \mathbf{g}^{-1} \mathbf{m}. \]  \hspace{1cm} (34)

We notice that matrix \( \mathbf{g} \) is a square matrix with the number of rows and columns equal to the number of constraints. Therefore, this method appears to be in this case preferable to that known as method of reduction by elements (see Appendix 1) since in practical cases the number of constraints results much smaller than the number of cross sections.

The estimators \( \tilde{\mathbf{y}} \) of \( \mathbf{y} \) take the form:

\[ \tilde{\mathbf{y}} = \mathbf{y}^{ex} + \tilde{\mathbf{v}}. \]  \hspace{1cm} (35)

For determining the dispersion matrix \( \mathbf{B}_{\tilde{\mathbf{y}}} \) of the random vector \( \tilde{\mathbf{y}} \), we recall that \( \mathbb{E}(\tilde{\mathbf{y}}) = \mathbf{y} \) and therefore

\[ \mathbf{B}_{\tilde{\mathbf{y}}} = \mathbb{E}[(\tilde{\mathbf{y}}-\mathbf{y})(\tilde{\mathbf{y}}-\mathbf{y})^T]. \]  \hspace{1cm} (36)

Using Eqn. (35) and (34), and recalling Eqns. (19) and (30) we find:
\[ \tilde{y} - y = -(U - B_y \hat{S}^T \hat{y}^{-1} \hat{S})y \] (37)

where \(U\) is a unit matrix. Since \(B_y = B_y\), as may be seen from Eqn. (19), we may write \((x)\):

\[ B_y = (U - B_y \hat{S}^T \hat{y}^{-1} \hat{S})B_y (U - B_y \hat{S}^T \hat{y}^{-1} \hat{S})^T. \] (38)

Now let us define the matrix:

\[ \mathcal{T} = B_y \hat{S}^T \hat{y}^{-1} \hat{S}. \] (39)

\((x)\) If \(y_{ex}\) has a dispersion matrix \(B_y\), it is simple to obtain the dispersion matrix \(B_x\) of a vector \(x_{ex}\) derived linearly from the relationship

\[ x = \mathcal{F} y \]

where \(\mathcal{F}\) represents a given matrix with elements \(f_{ij}\). In fact, recalling the relationships between expected values:

\[ E(a + b) = E(a) + E(b) \]
\[ E(ka) = kE(a), \quad (a = \text{constant}) \]

the \((i,j)\)-th element of \(B_x\) is given by:

\[ E[(x_i^{ex} - x_i)(x_j^{ex} - x_j)] = f_{ik} f_{jk} \sum_{k,h} E[(y_k^{ex} - y_k)(y_h^{ex} - y_h)] \]

which may be written in matrix form:

\[ B_x = \mathcal{F} B_y \mathcal{F}^T. \]
Remembering that \( \mathbf{g} = \mathbf{B}_y \mathbf{z}^T \), we see easily that

\[
\mathbf{z}^2 = \mathbf{z}
\]

\[
\mathbf{z}^T = \mathbf{B}_y^{-1} \mathbf{z} \mathbf{B}_y
\]

Eqn. (38) therefore may be written:

\[
\tilde{\mathbf{B}}_y = (\mathbf{u} - \mathbf{z}) \mathbf{B}_y (\mathbf{u} - \mathbf{z})^T
\]

\[
= (\mathbf{u} - \mathbf{z}) \mathbf{B}_y (\mathbf{u} - \mathbf{B}_y^{-1} \mathbf{z} \mathbf{B}_y)
\]

\[
= (\mathbf{B}_y - \mathbf{z} \mathbf{B}_y)\). \quad (42)
\]

2. \( \chi^2 \) Test

It may be shown (see Appendix 2) that the random quantity

\[
\tilde{\mathbf{R}} = \mathbf{v}^T \mathbf{B}_y^{-1} \mathbf{v}
\]

is distributed as \( \chi^2 \) with \( q \) degrees of freedom and therefore its expected value \( E(\tilde{\mathbf{R}}) \) is equal to the number \( q \) of constraints. According to this, given a cross section
library and an integral data set, we can test the assumptions made in order that the correlation procedure be valid. We recall them:

a) The errors of the cross sections represent random quantities normally distributed.

b) The errors of the integral data represent random quantities normally distributed.

c) No systematic errors are present in the given experimental and differential data and the dispersion matrices are adequate.

d) The quantities \( \tau_{J+i} \ (i=1, \ldots, I) \) (given by Eqn. (4) where, in practical cases, \( \tau_{i} \) are numerically assumed equal to \( \tau_{i}^{0} \)) are small values, so that second and higher order terms may be neglected.

e) The calculational model for the evaluation of the sensitivity coefficients \( s_{j,J+i} \ (j=1, \ldots, J; i=1, \ldots, I) \) is adequate.

The procedure to be adopted is standard, according to the current methods of hypothesis testing. Assumed "a priori" a probability, or confidence coefficient, \( p_{0} \) (for instance 0.9), we find a value \( \alpha_{q} \) such that the integral function

\[
K_{q}(\alpha_{q}) = \int_{0}^{\chi_{q}} \chi_{q}^{2}(x)dx = p_{0}
\]  \hspace{1cm} (44)

If the result obtained by this procedure shows that the value \( \hat{R} \) falls within the interval \((0, \chi_{q})\) and therefore, as is said in these cases, the probability falls in the \( p_{0} \)
level of significance, there are serious reasons indicating that the assumptions made are adequate. If, on the other hand, the value $\bar{N}$ falls out of the interval $(0, \alpha_q)$, and therefore the probability falls in the $(1-p_q)$ level of significance, there are serious reasons indicating that one or more of the assumptions made are not satisfied. According to specific judgment, this method of testing may be used effectively:

1- For indicating the probable presence, or not, of systematic errors and helping to sort them out.
2- For testing different cross section libraries, or pieces of libraries, to help an optimal selection.

3. - NON-NORMALITY OF ERRORS

A final comment seems appropriate regarding the possible non-normality of errors of the experimental data. In the present derivation the least square method was adopted, which may be considered as a consequence of introducing normal distributions in the likelihood function to be minimized. As well known, in this case the estimators $\bar{y}_i$ for a wide class of unbiased estimators result optimal in a certain sense, that is, for all indexes $i$: 
\[ P(|\tilde{y}_1 - y_1| < \varepsilon) \geq P(|y^*_1 - y_1| < \varepsilon) \] (45)

where \( y^*_1 \) represents a different unbiased estimator of \( y_1 \) and \( \varepsilon \) a small quantity. If the values \( y^*_1 \) are no more normally distributed, property (45) results no more generally valid, although the estimators \( \tilde{y}_1 \) still maintain a few important properties, namely:

1. They maintain their unbiasedness. This may be easily verified from Eqns. (35), (34), (21) and (13):

\[ E(\tilde{y}) = E(y^*_1 + \tilde{y}) = y \, . \]

2. They are characterized by minimal variance among all unbiased linear estimators in accordance with the theorem by Neyman and David (see Appendix 3).

3. For a wide number of independent experimental (differential and integral) data, they result asymptotically normally distributed, in accordance with Liapunov's theorem (see Appendix 4).

These properties are highly relevant to applications since they allow in general to say that:

a) By the least square procedure the estimators \( \tilde{y}_1 \) may be improved consistently with the improvement of the experimental information.

b) The integral parameters obtained by using the estimates \( \tilde{y}_1 \) may be assessed in terms of normal distribution and minimal variance.
APPENDIX 1. METHOD OF REDUCTION BY ELEMENTS.

We have to evaluate a vector \( \bar{\hat{y}} \) which satisfies the condition

\[
(\hat{y}^{ex} - \bar{\hat{y}})^T \Sigma_y^{-1} (\hat{y}^{ex} - \bar{\hat{y}}) = \text{Minimum} \quad (A1.1)
\]

and the fundamental equations

\[
\begin{align*}
& s_{j,0} + \sum_{i=1}^{N} s_{j,i} y_i = 0 \quad (j=1,2,\ldots,q) \\
& \quad (q < N)
\end{align*} 
\]

(with the true values \( y_i \) replaced by the estimates \( \hat{y}_i \)). By hypothesis, \( \text{rank}(\hat{S}) = q \), where \( \hat{S} \) represents the matrix of coefficients \( s_{j,i} \). Therefore, there exist \( (N-q) \) independent variables \( y_{1} \). Let us suppose they are indexed: \((q+1),(q+2),\ldots,N\). The first \( q \) variables result functions of the latter ones:

\[
y_j = s'_{j,0} + \sum_{i=1}^{N-q} s'_{j,i} y_{i+q} \quad (j=1,2,\ldots,q) \quad (A1.3)
\]
where $a'_{j,i}$ represent calculable coefficients. To these equations we may add the following identities:

$$y_j = \sum_{i=1}^{N-q} \delta_{ji} y_{i+q} \quad [i=(q+1), \ldots, N] \quad (A1.4)$$

where $\delta_{ji}$ is the Kroecker’s symbol.

If we interpret the parameters $y_i \quad [i=(q+1), \ldots, N]$ at the right hand side of Eqs. (A1.3) and (A1.4) as coefficients, or elements, $a_{i-1}$ and define the matrix

$$A' = \begin{bmatrix}
    s'_{1,1} & s'_{1,2} & \ldots & s'_{1,(N-q)} \\
    s'_{2,1} & s'_{2,2} & \ldots & s'_{2,(N-q)} \\
    \vdots & \vdots & \ddots & \vdots \\
    s'_{q,1} & s'_{q,2} & \ldots & s'_{q,(N-q)} \\
    1 & 0 & \ldots & 0 \\
    0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 1
\end{bmatrix} \quad (A1.5)$$

introducing the derived parameters:

\[\text{\textsuperscript{(*)}} (N-q) \text{ of them equivalent to } s'_{j,i} \text{ as given by Eqn. (5) with indexing shifted.}\]
\[ y_j' = \begin{cases} y_j - s_j' & \text{for } j = 1, 2, \ldots, q \\ y_j & \text{for } j = (q+1), \ldots, N \end{cases} \]  

(Eq. A1.6)

Eqs. (A1.3) and (A1.4) may be represented by the vector equation:

\[ \mathbf{y}' = \mathbf{G} \mathbf{a} \]  

(A1.7)

and our problem is transformed in that of evaluating an estimate \( \mathbf{\tilde{a}} \) of vector \( \mathbf{a} \), once the vector of experimental values \( \mathbf{y}^{\text{ex}} \) is known, such that

\[ (\mathbf{y}^{\text{ex}} - \mathbf{G} \mathbf{\tilde{a}})^T \mathbf{G}^{-1} (\mathbf{y}^{\text{ex}} - \mathbf{G} \mathbf{\tilde{a}}) = \text{Minimum} \]  

(A1.8)

The necessary conditions for a minimum are of the form:

\[ \mathbf{G}^T \mathbf{G}^{-1} (\mathbf{y}^{\text{ex}} - \mathbf{G} \mathbf{\tilde{a}}) = \mathbf{0} \]  

(A1.9)

which represents the so-called system of normal equations. From this we may write the solution:

\[ \mathbf{\tilde{a}} = \mathbf{G}^{-1} \mathbf{y}^{\text{ex}} \]  

(A1.10)

where \( \mathbf{G} \) represents the symmetric matrix

\[ \mathbf{G} = \mathbf{G}^T \mathbf{G}^{-1} \mathbf{G} \]  

(A1.11)

while the correlation matrix \( \mathbf{B}_a \) results:
\[ \mathcal{B}_\alpha = (\mathcal{C}^{-1} \mathcal{P}^T \mathcal{B}_\beta^{-1}) \mathcal{B}_\beta (\mathcal{C}^{-1} \mathcal{P}^T \mathcal{B}_\beta^{-1})^T \]
\[ = \mathcal{C}^{-1}. \quad \text{(A1.12)} \]

From Eqns. (A1.7) and (A1.6) we obtain the following expression for vector \( \tilde{y} \):
\[ \tilde{y} = \begin{vmatrix} (s_0^t \mathcal{L}_x) \\ \tilde{a} \end{vmatrix}, \quad \text{(A1.13)} \]

while the correlation matrix \( \mathcal{B}_\gamma \) results the following:
\[ \mathcal{B}_\gamma = \mathcal{P} \mathcal{B}_\alpha \mathcal{P}^T = \mathcal{P} \mathcal{C}^{-1} \mathcal{P}^T. \quad \text{(A1.14)} \]

Since conditions (A1.8) and (A1.7) result equivalent to conditions (22) and (23), the two methods considered appear equivalent as far as the results are concerned. This may be also shown by expressing the resulting Eqns. (35) and (A1.13) in terms of the elements \( \mathcal{S}' \), \( \mathcal{U} \), \( \mathcal{B}_Q \), \( \mathcal{B}_\sigma \) of the partitioned matrices \( \mathcal{S}' \), \( \mathcal{B}_\gamma \) (and, consequently, \( \mathcal{C} \) and \( \tilde{y} \)) and of the components \( \mathcal{L}_x^{\text{ex}} \) and \( y_Q^{\text{ex}} \) of vector \( \mathcal{Y}^{\text{ex}} \). Setting in Eqn. (2), and following ones, for simplicity, \( \sigma_{o,i} = \sigma_{i}^{\text{ex}} \) (and, therefore, \( y_o^{\text{ex}} = 0 \)), Eqn. (A1.13) [or, more simply, the component \( \tilde{y}_o = \tilde{a} \), with \( \tilde{a} \) given by Eqn. (A1.10)], relevant to the method by elements, becomes
\[ \tilde{y} = (\mathcal{S}'^T \mathcal{B}_Q^{-1} \mathcal{S}' + \mathcal{B}_\sigma^{-1})^{-1} \mathcal{S}'^T \mathcal{B}_Q^{-1} y_Q^{\text{ex}}. \quad \text{(A1.15)} \]
which may be written

\[(\gamma' T \beta_Q^{-1} \gamma' + \beta_Q^{-1})\tilde{\gamma}_o = \gamma' T \beta_Q^{-1} \tilde{y}_Q^e\]  \hspace{1cm} (A1.16)

i.e.: \[\tilde{\gamma}_o = \beta_Q^{-1} \gamma' T \beta_Q^{-1} (y_Q^e - \gamma' \tilde{\gamma}_o)\] \hspace{1cm} (A1.17)

Setting:

\[x = \beta_Q^{-1} (y_Q^e - \gamma' \tilde{\gamma}_o)\] \hspace{1cm} (A1.18)

we obtain:

\[\tilde{\gamma}_x = \beta_Q^{-1} x^T\] \hspace{1cm} (A1.19)

and substituting this expression in Eqn. (A1.16):

\[(\gamma' T \beta_Q^{-1} \gamma' \beta_Q + \gamma' \gamma) \gamma' T x = \gamma' T \beta_Q^{-1} y_Q^e\] \hspace{1cm} (A1.20)

Multiplying on the left by \((\gamma' \gamma)^{-1} \gamma'\) we obtain:

\[(\gamma' \beta_Q^{-1} \gamma' T + \beta_Q) x = y_Q^e\] \hspace{1cm} (A1.21)

from which, substituting in Eqn. (A1.19), it results:

\[\tilde{\gamma}_o = \beta_Q^{-1} \gamma' T (\gamma' \beta_Q^{-1} \gamma' T + \beta_Q) y_Q^e\] \hspace{1cm} (A1.22)

which coincides with the expression (35) relevant to the
method by the Lagrange multipliers when the component vector \( \tilde{y}_p \) is considered separately. Since it results \( \tilde{x}_Q = S' \tilde{y}_p \), the equivalence between the two methods is shown also for \( \tilde{x}_Q \).
APPENDIX 2. DISTRIBUTION OF $\tilde{r}$.

Since

$$m = \bar{a}_0 + \tilde{f}_y^e x$$  \hspace{1cm} (A2.1)

$$\tilde{v} = \tilde{f}_y^e + \tilde{y}$$  \hspace{1cm} (A2.2)

$$\tilde{y} = -\beta_y \tilde{f}_y^e \tilde{f}_y^{-1} m$$  \hspace{1cm} (A2.3)

we can write:

$$\tilde{v} - \tilde{f}_y^e - \beta_y \tilde{f}_y^e \tilde{f}_y^{-1} m$$

$$= \tilde{f}_y^e - \beta_y \tilde{f}_y^e \tilde{f}_y^{-1} (\bar{a}_0 + \tilde{f}_y^e x)$$

$$= -\beta_y \tilde{f}_y^e \tilde{f}_y^{-1} \bar{a}_0 + \left[ \tilde{U} - \beta_y \tilde{f}_y^e \tilde{f}_y^{-1} \tilde{f}_y \right] \tilde{y}^e$$  \hspace{1cm} (A2.4)

and therefore:

$$\tilde{v} = \tilde{v} - \tilde{f}_y^e = -\beta_y \tilde{f}_y^e \tilde{f}_y^{-1} (\bar{a}_0 + \tilde{f}_y^e x)$$  \hspace{1cm} (A2.5)
Since
\[ \mathbf{v} = y - y^e \]  \hspace{1cm} (A2.6)
we have
\[ \mathbf{E} \mathbf{v} = 0 \] \hspace{1cm} (A2.7)
\[ \mathbf{B}_y = \mathbf{B}_y = \mathbf{B}_y \] \hspace{1cm} (A2.8)

Using Eqn. (A2.5) and recalling that
\[ \mathbf{s}_0 + \mathbf{f}_y = 0 \] \hspace{1cm} (A2.9)
Eqn. (A2.5) may be written:
\[
\tilde{\mathbf{v}} = -\mathbf{B}_y \mathbf{S}^T \mathbf{f}_y^{-1} [\mathbf{s}_0 + \mathbf{f}_y (y - \mathbf{v})]
\]
\[ = \mathcal{Z}_y \mathbf{v} \] \hspace{1cm} (A2.10)
where
\[ \mathcal{Z} = \mathbf{B}_y \mathbf{S}^T \mathbf{f}_y^{-1} \mathbf{f}_y \] \hspace{1cm} (A2.11)

We make now the transformation
\[ \mathbf{v}^0 = \frac{1}{\mathbf{B}_y} \mathbf{v} \] \hspace{1cm} (A2.12)

We can easily see, from Eqns. (A2.6) and (A2.8), that:
\[
\begin{align*}
\bar{y}^o \ &= \ \hat{B}_y^{-\frac{1}{2}} \bar{y} = 0 \\
\hat{B}_v^o \ &= \ \hat{B}_y^{-\frac{1}{2}} \hat{B}_v \hat{B}_y^{-\frac{1}{2}} = \mathcal{U}
\end{align*}
\]

where \(\mathcal{U}\) represents a unit matrix.

Replacing vector \(v\) in Eqn. (A2.10) with its expression obtainable from Eqn. (A2.12), the former becomes:

\[
\bar{y} = \hat{B}_y^{\frac{1}{2}} \bar{y}^o
\]

and the quadratic random quantity

\[
\tilde{R} = \bar{y}^T \hat{B}_y^{-1} \bar{y}
\]

may be written, recalling the properties (40) and (41):

\[
\tilde{R} = \bar{y}^o \hat{B}_y^{\frac{1}{2}} \bar{T} \hat{B}_y^{-1} \bar{T} \hat{B}_y^{\frac{1}{2}} \bar{y}^o
\]

\[
\begin{align*}
= \bar{y}^o \hat{B}_y^{\frac{1}{2}} \bar{T} \hat{B}_y^{-1} \bar{T} \hat{B}_y^{\frac{1}{2}} \bar{y}^o \\
= \bar{y}^o \hat{B}_y^{-\frac{1}{2}} \bar{T} \hat{B}_y^{\frac{1}{2}} \bar{y}^o \\
= \bar{y}^o \hat{B}_y^{-\frac{1}{2}} \bar{T} \hat{B}_y^{\frac{1}{2}} \bar{y}^o
\end{align*}
\]

Introducing the matrix

\[
\mathcal{W} = \hat{B}_y^{-\frac{1}{2}} \bar{T} \hat{B}_y^{\frac{1}{2}}
\]

(A2.18)
Eqn. (A2.17) becomes:

\[ \tilde{R} = \mathbf{v}^T \mathbf{W} \mathbf{v} \]  

(A2.19)

Recalling Eqns. (40) and (41) we see immediately that

\[ \mathbf{W}^T = \mathbf{W} \]  

(A2.20)

\[ \mathbf{W}^2 = \mathbf{W} \]  

(A2.21)

Since the matrix \( \mathbf{W} \) is symmetrical, we can find an orthogonal transformation

\[ \mathbf{z} = \mathbf{f} \mathbf{y} \]  

(A2.22)

such that the quadratic form (A2.19) becomes:

\[ \tilde{R} = \mathbf{z}^T \mathbf{D} \mathbf{z} \]  

(A2.23)

where \( \mathbf{D} \) represents the diagonal matrix

\[
\mathbf{D} = \begin{bmatrix}
  d_1 & 0 & \cdots & 0 \\
  0 & d_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_N \\
\end{bmatrix} = \mathbf{f}^T \mathbf{W} \mathbf{f}  
\]

(A2.24)

Since, from the property of the orthogonal matrices,
\[ Z^T = Z^{-1} , \]  
(A2.25)

we see immediately, recalling Eqn. (A2.21), that

\[ D^2 = D . \]  
(A2.26)

This means that the elements \( d_i \) may be either equal to 1 or 0, i.e.

\[ d_i = \begin{cases} 1 \\ 0 \end{cases} . \]  
(A2.27)

We may also assess that the rank of matrix \( D \) is equal to the number of units among the values \( d_i \). Recalling the general relation valid for a product of matrices \( \hat{A}_1 \hat{A}_2 \ldots \hat{A}_k \):

\[
\text{rank}(\hat{A}_1 \hat{A}_2 \ldots \hat{A}_k) \leq \\
\leq \min\left[\text{rank}(\hat{A}_1), \text{rank}(\hat{A}_2), \ldots, \text{rank}(\hat{A}_k)\right]
\]  
(A2.28)

we can show that \( \text{rank}(Z) \) is equal to the number of constraints \( q \). In fact, having assumed the linear independence of the constraints, we can write:

\[ \text{rank}(Z) = q \]  
(A2.29)
\[ \text{rank}(\hat{G}^{-1}) = \text{rank}(\hat{G}) \]
\[ = \text{rank}(\hat{S}\hat{\beta}_y \hat{S}^T) \leq q \]
(A2.30)

and therefore:
\[ \text{rank}(\hat{\Sigma}) = \text{rank}(\hat{\beta}_y \hat{S}^T \hat{G}^{-1} \hat{S}) \leq q \]
(A2.31)

On the other hand, since, recalling Eqn. (40)
\[ \hat{S}\hat{\Sigma} = \hat{S}\hat{\Sigma}^{-1}\hat{\Sigma}^2 = \hat{S}\hat{\Sigma}^{-1}\hat{\Sigma} = \hat{S} \]
(A2.32)

we may write the relation
\[ \text{rank}(\hat{\Sigma}) \geq q \]
(A2.33)

which, together with condition (A2.31), imposes the equation:
\[ \text{rank}(\hat{\Sigma}) = q \]
(A2.34)

Now, recalling relations (40) and (A2.21), we may write the equations:
\[ \mathbf{w}D = \mathbf{w}; \quad D\mathbf{w} = D \]
(A2.35)
\[ \hat{\Sigma}\mathbf{w} = \hat{\Sigma}; \quad \mathbf{w}\hat{\Sigma} = \mathbf{w} \]
(A2.36)

from which we obtain the conditions:
\[ \text{rank}(\mathcal{W}) \leq \text{rank}(\mathcal{D}) \quad \text{and} \quad \text{rank}(\mathcal{Z}) \leq \text{rank}(\mathcal{W}) \quad (A2.37) \]

\[ \text{rank}(\mathcal{Z}) \leq \text{rank}(\mathcal{W}) \quad \text{and} \quad \text{rank}(\mathcal{W}) \leq \text{rank}(\mathcal{Z}) \quad (A2.38) \]

The only possible solution results:

\[ \text{rank}(\mathcal{D}) = \text{rank}(\mathcal{W}) = \text{rank}(\mathcal{Z}) = q \quad (A2.39) \]

This allows us to write matrix \( \mathcal{D} \) in the form:

\[ \mathcal{D} = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\end{bmatrix} \quad (A2.40) \]

where, for practicality, we have chosen the first \( q \) values \( d_i = 1 \). Eqn. (A2.23) therefore may be written:

\[ \tilde{R} = \sum_{i=1}^{q} \frac{a_i^2}{a_i^2} \quad (A2.41) \]

From Fisher's theorem we know that, by an arbitrary orthogonal
al transformation, a random vector of $n$ independent variables equally distributed may be transformed into a vector of $n$ dimensions with the same distribution of the former one. Here the original random vector $\mathbf{v}^0$ exhibits properties (A2.13) and (A2.14) and therefore the same hold also for vector $\mathbf{z}$ given by the orthogonal transformation (A2.22). As a consequence, the quadratic form (A2.41) represents a sum of $q$ terms normally distributed with zero mean and unit variance, that is, it represents a $\chi^2_q$ distribution.
APPENDIX 3. MINIMAL VARIANCE OF $y_j$ IN NON-NORMAL DISTRIBUTIONS.

We shall demonstrate the minimal variance of estimators $\tilde{y}_j$ which appear in the method using elements (see Appendix 1). Since $\tilde{y}_j$ are given by the linear vector relationship (A1.13):

$$
\begin{bmatrix}
\tilde{y} \\
\tilde{\alpha}
\end{bmatrix} = 
\begin{bmatrix}
(\mathbb{a}_0' + \tilde{y}'(\mathbb{B}^{-1}_y)^{c_0}) \\
\mathbb{C}^{-1}_y (\mathbb{P}_y^{-1} \mathbb{B}^{-1}_y)^{c_0}
\end{bmatrix}
$$

$$= 
\begin{bmatrix}
(\mathbb{a}_0' + \tilde{y}'\tilde{\alpha}) \\
\tilde{\alpha}
\end{bmatrix} \tag{A2.1}
$$

[which we have shown to be equivalent to expression (35)], it is clear that the minimal variance for the estimators $\tilde{y}_j$ will follow.

Let us consider a generic parameter $a_i$. If we define the vector:

...
\[ \mathbf{u}_i = \begin{bmatrix} 0,0,\ldots,0,1,0,\ldots,0 \end{bmatrix} \quad \text{(i columns)} \]  

the following identity may be written:

\[ \mathbf{a}_i = \mathbf{g}_i \mathbf{a} \]  

(A3.3)

and the variance \( \text{D}(\mathbf{a}_1) \) will be given by the expression:

\[ \text{D}(\mathbf{a}_1) = \mathbf{g}_i \mathbf{\Omega}_\mathbf{a} \mathbf{g}_i^T \]  

(A3.4)

where \( \mathbf{\Omega}_\mathbf{a} \) represents the correlation matrix of vector \( \tilde{\mathbf{a}} \) and is given by Eqn. (A1.12), i.e.:

\[ \mathbf{\Omega}_\mathbf{a} = \mathbf{C}^{-1} = \left( \mathbf{P}^T \mathbf{\Omega}_y^{-1} \mathbf{P} \right)^{-1} \]  

(A3.5)

Now let us consider another linear unbiased estimator \( \mathbf{a}_1' \) of \( \mathbf{a}_1 \) of the form:

\[ \mathbf{a}_1' = l_{1,1}\mathbf{y}_1^{\text{ex}} + l_{1,2}\mathbf{y}_2^{\text{ex}} + \ldots + l_{1,N}\mathbf{y}_N^{\text{ex}} \]  

(A3.6)

if we define the vector

\[ \mathbf{l}_i = \begin{bmatrix} l_{i,1} \quad l_{i,2} \quad \ldots \quad l_{i,N} \end{bmatrix} \]  

(A3.7)

Eqn. (A3.6) may be synthetically written:

\[ \mathbf{a}_1' = \mathbf{l}_i \mathbf{y}^{\text{ex}} \]  

(A3.8)
Because of its unbiasedness, it must be, recalling Eqn. (A3.3),

\[
E(a'_1) = a_1 = \frac{1}{n} E(y_{\text{ex}}) = \frac{1}{n} y' = \frac{1}{n} \mathbf{C} a = \mathbf{g}_1 a = \mathbf{g}_1 .
\]  \hspace{1cm} (A3.9)

Let us for a moment consider those cases where the parameters \(a_1\) result independent from each other (recalling their definition, this corresponds to assuming that the last \((N-q)\) rows and columns of the correlation matrix \(\mathbf{B}_y\) have a diagonal form). In this case, the relationship (A3.9) may be true only if

\[
\frac{1}{n} \mathbf{C} = \mathbf{g}_1 .
\]  \hspace{1cm} (A3.10)

Now our assumption will be proved for these cases if we can show that the variance

\[
\mathbf{D}(a'_1) \geq \mathbf{D}(\tilde{a}_1) \ , \hspace{1cm} (A3.11)
\]

or, recalling expressions (A3.8) and (A3.4), that

\[
\frac{1}{n} \mathbf{B}_1 - \frac{1}{n} \mathbf{a} \mathbf{a}^T \geq \mathbf{g}_1 \mathbf{B}_y^{-1} \mathbf{g}_1 . \hspace{1cm} (A3.12)
\]

Recalling Eqn. (A3.10) and expression (A3.5) of the correlation matrix \(\mathbf{B}_y\), this relationship becomes:

\[
\frac{1}{n} \mathbf{B}_y^{-1} - \frac{1}{n} \mathbf{a} \mathbf{a}^T \geq \frac{1}{n} \mathbf{g} (\mathbf{C}^T \mathbf{B}_y^{-1} \mathbf{C})^{-1} \mathbf{C}^T \frac{1}{n} \mathbf{a} . \hspace{1cm} (A3.13)
\]
If we define the matrices
\[ \zeta = \mathcal{P} (\mathcal{P}^T \mathcal{B}_y^{-1} \mathcal{P})^{-1} \mathcal{P}^T \]
\[ \eta = \mathcal{P} (\mathcal{P}^T \mathcal{B}_y^{-1} \mathcal{P})^{-1} \mathcal{P}^T \mathcal{B}_y^{-1} \]
we may write:
\[ \eta^2 = \eta \] \hspace{1cm} (A3.15)
\[ \zeta^T = \zeta \] \hspace{1cm} (A3.16)
\[ \zeta = \eta \mathcal{B}_y \] \hspace{1cm} (A3.17)
and the right hand side of Eqn.(A3.13) may be written
\[ \frac{1}{l_1} \mathcal{P} (\mathcal{P}^T \mathcal{B}_y^{-1} \mathcal{P})^{-1} \mathcal{P}^T \frac{1}{l_1} = \frac{1}{l_1} \zeta \frac{1}{l_1} \] \hspace{1cm} (A3.18)
With the transformation
\[ \frac{1}{l_1} \mathcal{B}_y^{\frac{1}{2}} = m \] \hspace{1cm} (A3.19)
Eqn.(A3.18) becomes:
\[ \frac{1}{l_1} \zeta \frac{1}{l_1} = m \mathcal{B}_y^{-\frac{1}{2}} \zeta \mathcal{B}_y^{-\frac{1}{2}} m^T \] \hspace{1cm} (A3.20)
Since the matrix \((\mathcal{B}_y^{-\frac{1}{2}} \zeta \mathcal{B}_y^{-\frac{1}{2}})\) is symmetric and since, from Eqn.(A3.17), we have:
$$\beta_y^{-\frac{1}{2}} Z \beta_y^{-\frac{1}{2}} = \beta_y^{-\frac{1}{2}} U \beta_y^{-\frac{1}{2}}$$  \hspace{1cm} (A3.21)$$

and therefore, recalling property (A3.15),

$$\left(\beta_y^{-\frac{1}{2}} Z \beta_y^{-\frac{1}{2}}\right)^2 = \beta_y^{-\frac{1}{2}} U^2 \beta_y^{-\frac{1}{2}} = \beta_y^{-\frac{1}{2}} Z \beta_y^{-\frac{1}{2}}$$  \hspace{1cm} (A3.22)$$

we may find an orthogonal transformation

$$\tilde{x} = M \tilde{y}$$  \hspace{1cm} (A3.23)$$

where matrix $M$ is such that

$$M^T \beta_y^{-\frac{1}{2}} Z \beta_y^{-\frac{1}{2}} M = \mathcal{D}$$  \hspace{1cm} (A3.24)$$

$\mathcal{D}$ being a diagonal matrix, i.e.

$$\mathcal{D} = \begin{bmatrix}
d_1 & 0 & \ldots & 0 \\
0 & d_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & d_N
\end{bmatrix}$$  \hspace{1cm} (A3.25)$$

Recalling that $M^T = M^{-1}$, it is easy to see that

$$\mathcal{D}^2 = \mathcal{D}$$  \hspace{1cm} (A3.26)$$

Consequently, also $d_1^2 = d_1$, which means that the elements
\[ d_i = 0 \text{ or } 1. \text{ The quadratic form (A3.22) then becomes} \]
\[
\frac{1}{2} \mathcal{F}^T \mathcal{B}_y^{-\frac{1}{2}} \mathcal{Z} \mathcal{B}_y^{-\frac{1}{2}} \mathcal{F}_t^T = \frac{1}{2} \mathcal{D}_i = \\
= \sum_{i=1}^{N} a_i t_i^2 . \quad (d_i = 0 \text{ or } 1) \quad (A3.27)
\]

We now reduce the left hand side of relation (A3.12) into the form:
\[
1 \mathcal{B}_y^{-\frac{1}{2}} \mathcal{F}_m^T = \mathcal{F}_m^T = \\
= \mathcal{F} \mathcal{F}_m^T = \sum_{i=1}^{N} t_i^2 . \quad (A3.28)
\]

Since
\[
D(a_1') = \frac{1}{2} \mathcal{B}_y^{-\frac{1}{2}} = \sum_{i=1}^{N} t_i^2 \leq \sum_{i=1}^{N} a_i t_i^2 = \\
= \frac{1}{2} \mathcal{F}^T (\mathcal{P}_y^T \mathcal{B}_y^{-1} \mathcal{P})^{-1} \mathcal{P}_t^T \mathcal{D}_i = g_{\frac{1}{2}} \mathcal{B}_y g_{\frac{1}{2}}^T = D(\tilde{a}_1) , \quad (A3.29)
\]

the minimal variance for the estimator \( \tilde{a}_1 \) is proved for those cases in which they are supposed independent from each other. Let us now suppose that the parameters \( \tilde{a}_1 \) are correlated with each other (which results from non-zero off-diagonal elements in the last \((N-q)\) rows and columns of the correla-
tion matrix \( \beta_y \). In order to extend the demonstration also to this case, we consider the quadratic form

\[
R = (y^{ex} - \bar{y})^T \beta_y^{-1} (y^{ex} - \bar{y})
\]

which appears at the exponent of the likelihood function (16). The least square methods are based on minimizing this quadratic form so that estimators \( \hat{\bar{y}} \) of \( \bar{y} \) are found which satisfy given constraints and the condition:

\[
\hat{R} = (y^{ex} - \hat{\bar{y}})^T \beta_y^{-1} (y^{ex} - \hat{\bar{y}}) = \text{Minimum} \quad (A3.31)
\]

Since \( \beta_y^{-1} \) is a symmetric matrix, an orthogonal transformation

\[
u = \tilde{\mathcal{F}} y
\]

(A3.32)

may be found such that the quadratic form (A3.30) becomes (indicating by \( u^{ex} \) the derived quantity \( \tilde{\mathcal{F}} y^{ex} \)):

\[
R = (u^{ex} - \bar{u}) \beta_u^{-1} (u^{ex} - \bar{u})
\]

(A3.33)

where

\[
\beta_u = \tilde{\mathcal{F}} \beta_y^{-1} \tilde{\mathcal{F}}^T
\]

(A3.34)

represents a diagonal correlation matrix, with the number of elements equal to that of the independent variables (which means that, in general, vector \( u \) may have one or more null
components). In correspondence with this transformation, the fundamental equations (18) become:

\[ \hat{z}_0 + \sum \mathcal{F}^T u = 0 \quad \text{(A3.35)} \]

Following then the reduction by elements, we fall in the category of cases with independent parameters (elements) treated previously. Since \( \tilde{a}_i = \tilde{y}_{i+q} \) and \( \tilde{y} = \mathcal{F}^T \tilde{u} \), the general conclusion of the mininumness of the variance of the estimators \( \hat{a}_i \) for all the cases which may be encountered results proved.
APPENDIX 4. ASYMPTOTIC NORMAL DISTRIBUTION OF $\tilde{y}_1$.

If we consider a sum of variables independent from each other

$$\tilde{S}_n = \tilde{h}_1 + \tilde{h}_2 + \ldots + \tilde{h}_n \quad (A4.1)$$

with expected values

$$E(\tilde{h}_1) = h_1 \quad (A4.2)$$

and moments

$$E(|\tilde{h}_1 - h_1|^2) = d_1 \quad (variance) \quad (A4.3)$$

$$E(|\tilde{h}_1 - h_1|^3) = g_1, \quad (A4.4)$$

Liapunov's theorem states that $\tilde{S}_n$ is asymptotically normal if the ratio
\[ \sum_{i=1}^{n} \frac{\varepsilon_i}{(\sum_{i=1}^{n} d_i)^{3/2}} \rightarrow 0 \quad \text{for} \quad n \rightarrow \infty , \]  

i.e. if no term in the sum results dominating on the others.

Let us now consider the expression of \( \tilde{\mathbf{y}}' \) obtainable from Eqn. (35):

\[ \tilde{\mathbf{y}} = \mathbf{y}^{\text{ex}} + \tilde{\mathbf{y}} = q + \mathbf{2}_{y}^{\text{ex}} \]  

where \( q \) and \( \mathbf{2} \) are a given constant vector and matrix respectively, which here need not be defined. Since matrix \( \mathbf{B}_y \) is symmetric, we may determine an orthogonal transformation

\[ \mathbf{z} = \mathbf{\xi}^{\text{ex}} \]  

such that the correlation matrix

\[ \mathbf{B}_z = \mathbf{\xi} \mathbf{B}_y \mathbf{\xi}^T \]  

is diagonal. This amounts to say that the non-null components of vector \( \mathbf{z} \) represent independent variables. Eqn. (A4.6) may then be written:

\[ \tilde{\mathbf{y}} - q = \mathbf{2} \mathbf{\xi}^T \mathbf{z} . \]  

In vectorial form, this expression represents sums \( (\tilde{\mathbf{y}}_i - q_i) \)
of independent variables \((x_{ij} z_j)\) (where \(x_{ij}\) represent given constant coefficients) none of which is supposed to prevail on the others (as seems the case where a high enough number of independent cross section parameters result implied in the correlation procedure concerning, for example, capture, fission and downscattering events for a number of different materials). Consequently, it seems quite reasonable to say that a more or less marked tendency of the estimators \(\tilde{y}_i\) of assuming a normal distribution exists (in the presence of presumed non-normal distributions of the experimental data) in relation with the number and more or less distributed influence of independent cross section parameters involved in the correlation (and, of course, in relation with the number of integral measurements considered).

REFERENCES