Nuclear Data and Integral Measurements
Correlation for Fast Reactors.
Part 2: Review of Methods

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FOREWORD

This second part consists of a critical review and comments of a few methods proposed for cross section adjustment with integral measurements on fast reactors. The methods selected, although not all-inclusive, represent important more or less recent contributions on this field and all of them have in common a statistically-oriented approach. As a consequence of this common denominator, the intercomparisons result greatly simplified.

Because of their relevance, emphasis has been given in particular to those arguments concerning the correlation among the group cross sections and the presence of systematic errors.

Since the present work represents the continuation of the first part (see Ref. [1]), direct references to formulations to be found in this latter one have been made. These references are distinguished by a roman I [for example: Eqn. (15/I)].
1. PAZY, RAKAVY, REISS AND YELVIT'S METHOD [2]

The method proposed by Pazy et al. represents the first attempt to base the correlation procedure on a strict statistical basis. In their model cross sections were first considered as continuous functions of energy and then a discrete (group-wise) formulation was derived by imposing that the resulting adjusted cross sections have the form:

$$\tilde{\sigma}(E) = (1 + \tilde{p}_g) \sigma^{ex}(E)$$

(1.1)

where $\tilde{p}_g$ represents a parameter, constant within each energy group, to be estimated. The resulting expressions of the estimates and their standard deviations correspond to those obtained using the Lagrange multipliers. All the dispersion matrices relative to the experimental values appear to have a diagonal form, no correlation among different cross sections having being considered. Similarly, the existing

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(m) Except a correlation-equivalent introduction into the the likelihood function of the experimental value of the total cross section as a sum of the fission, capture, inelastic and elastic scattering contributions.
(i.e. depending on the fit) correlations among the adjusted cross sections appear neglected so that an incomplete error evaluation is obtained in the estimate of new integral parameters.

2. - DRAGT'S METHOD [3]

The interesting method suggested by Dragt follows that suggested by Rev. Thomas Bayes in a famous paper "An assay toward solving a problem in the doctrine of chances" published posthumously in 1763. It was specifically aimed at solving problems where the concept of probability could be used to describe our degree of belief that an unknown (fixed) parameter has a certain value (or set of values, or distribution), this representing what was called "prior knowledge" to be incorporated into the analysis of an experiment, but it may be successfully used also in different cases, with random variables in place of parameters. In these cases it represents an alternative way of using the maximum likelihood method, as will be shown later on.

Following Bayes' theorem, given a normal (prior) distribution \( P(\mathbf{y} \mid \mathbf{y}_{\text{ex}}) \) of parameters (for instance, in our case, relative cross section differences as defined by Eqn.(4/1)) represented by vector \( \mathbf{y} \) with mean value \( \mathbf{y}_{\text{ex}} \) and with dispersion matrix \( \mathbf{B}_{y} \) and given experimental values of integral parameters (in our case defined by Eqn.(1/1)) represented by vector \( \mathbf{y}_{\text{ex}} \) with covariance matrix \( \mathbf{C}_{\mathbf{y}} \), the posterior distribution \( P(\mathbf{y} \mid \mathbf{y}_{\text{ex}}, \mathbf{y}_{\text{ex}}) \) of
parameters $y^*$ is given by the equation:

$$P(y^* | y_e, y^*_Q) = \frac{P(y^* | y_e) L(y^* | y^*_Q)}{\int P(y^* | y_e) L(y^* | y^*_Q) dy^*}$$

(1.2)

where $L$ represents the likelihood function.

Since the error distributions are assumed normal, the likelihood function $L(y^* | y^*_Q)$ is given by the expression

$$L(y^* | y^*_Q) = a_1 \exp \left[ -\frac{1}{2} (y^* - y^*_Q)^T \delta Q^{-1} (y^* - y^*_Q) \right]$$

(3.2)

[*] It is simple to see how Eqn. (1.2) may be obtained from the following relation between conditional probabilities:

$$P(y^* | y^*_{Q}, y^*_Q) = P(y^* | y^*_Q) P(y^*_Q | y^*_{Q})$$

$$= P(y^*_Q | y^*_Q) P(y^* | y^*_Q, y^*_Q)$$

(2.2)

and recalling, since $L(a|b) = P(b|a)$, that

$$P(y^*_Q) = \int P(y^*_Q | y^*_Q) dy^*_Q = \int P(y^*_Q | y^*_Q) L(y^*_Q | y^*_Q) dy^*_Q$$

[In Eqn. (2.2) the prior values $y^*_Q$ are introduced, when pertinent, in order that the formalism be correct. It results obvious that the inconsistent form $P(y^*_Q | y^*, y^*_Q)$, in the middle term, should reduce to $P(y^*_Q | y^*)$, according with the conditional probability $P(y^* | y^*_Q)$].

where $a_1$ represents a constant coefficient while $y^*_Q$ results a function of $y^*$ since its components are represented by Eqn. (3.1) in which $Q_{ij}$ are given by Eqn. (1/1) (or Eqn. (2/1)). Consequently, Eqn. (1.2) becomes:

$$a_2 \exp \left[ -\frac{1}{2} (y^* - y^*_Q)^T \delta Q^{-1} (y^* - y^*_Q) \right] =$$

$$= a_3 \exp \left[ -\frac{1}{2} (y^* - y^*_Q)^T \delta Q^{-1} (y^* - y^*_Q) - \frac{1}{2} (y^*_Q - y^*_Q)^T \delta Q^{-1} (y^*_Q - y^*_Q) \right]$$

(4.2)

where $a_2$ and $a_3$ represent constant coefficients, while $y^*_Q$ represents an estimator of the mean value of the random vector $y^*$. Estimators $y^*_Q$ are obtained by equating the exponents of this equation or, more precisely, the coefficients of linear and quadratic terms in the variable $y^*$. It is simple to realize that finding the estimators $y^*_Q$ which appear in the (posterior) distribution at the left hand side of Eqn. (4.2) is equivalent to searching for $y^*_Q$ a vector which maximizes the term at its right hand side. Since this term corresponds to the likelihood function (16/1), which is extended to all variables (cross sections and integral data, correlated to each other by Eqn. (18/1) or by the equivalent Eqn. (A1.7/I)) the identity between the maximum likelihood method and that resulting from Bayes' theorem is apparent.

Following now the derivation along Bayes' theorem, equating the exponents of Eqn. (4.2) and expliciting the linear dependence of vector $y^*_Q$ from $y^*$ through the sensitivity matrix $S'$ defined in Eqn. (A1.5/I), we obtain, with some rearrangement:
where \( \mathcal{A}_4 \) represents a constant value, while \( \tilde{y}_Q \) is a vector obtained by setting \( y_r = \bar{y}_r^{ex} \) in the vector function \( y_Q(y_r) \).

Equating then the coefficients of linear and quadratic terms of \( y_r \), the following equations are derived:

\[
\begin{align*}
\beta_{\sigma}^{-1} \bar{y}_r &= (\beta_{\sigma}^{-1} + \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta') y_r^{ex} + \beta_{\sigma}' \mathcal{B}_Q^{-1} (\bar{y}_Q - \bar{y}_Q) \\
\beta_{\sigma}^{-1} &= \beta_{\sigma}^{-1} + \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta'.
\end{align*}
\]

(6.2) \hspace{1cm} (7.2)

from which vector \( \tilde{y}_r \) and the dispersion matrix \( \beta_{\sigma} \) may be determined. Substituting Eqn.(7.2) in Eqn.(6.2) we obtain:

\[
(\beta_{\sigma}^{-1} + \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta') (\bar{y}_Q - \bar{y}_Q) = \beta_{\sigma}' \mathcal{B}_Q^{-1} (\bar{y}_Q - \bar{y}_Q)
\]

(8.2)

\[
\beta_{\sigma}^{-1} = \beta_{\sigma}^{-1} + \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta'.
\]

(9.2)

Eqn.(8.2) may be written:

\[
\tilde{y}_r - y_r^{ex} = \beta_{\sigma} \beta_{\sigma}' \tilde{y}_r
\]

(10.2)

\[
x = \beta_{\sigma}^{-1} [\bar{y}_Q - \bar{y}_Q] - \beta_{\sigma}' \tilde{y}_r
\]

(11.2)

Substituting Eqn.(10.2) in Eqn.(8.2), multiplying on the left by \( \beta_{\sigma}^{-1} \beta_{\sigma}' \), (where the square matrix \( \beta_{\sigma} \beta_{\sigma}' \) is supposed to be non-singular), we obtain the system:

\[
\begin{cases}
x = (\mathcal{B}_Q + \mathcal{N})^{-1} (\bar{y}_Q - \bar{y}_Q) \\
\tilde{y}_r - y_r^{ex} = \beta_{\sigma} \beta_{\sigma}' \tilde{y}_r
\end{cases}
\]

(12.2) \hspace{1cm} (13.2)

where

\[
\mathcal{N} = \beta_{\sigma} \beta_{\sigma}' \beta_{\sigma}'
\]

(14.2)

For what concerns Eqn.(9.2), we find with some algebra:

\[
\mathcal{U} = \beta_{\sigma} \beta_{\sigma}' + \beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta'
\]

\[
\beta_{\sigma} = \beta_{\sigma} (\mathcal{U} + \beta_{\sigma}' \mathcal{B}_Q^{-1} \beta'_y \beta_y)
\]

\[
\beta_{\sigma} \beta_{\sigma}' = \beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q^{-1} (\beta_{\sigma} + \mathcal{N})
\]

\[
\beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q = \beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q \mathcal{N}
\]

\[
\beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q (\mathcal{N} + \mathcal{B}_Q) = \beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q (\mathcal{N} + \mathcal{B}_Q) = \beta_{\sigma} \beta_{\sigma}' \mathcal{B}_Q (\mathcal{N} + \mathcal{B}_Q)
\]

\[
\beta_{\sigma} \beta_{\sigma}' = \beta_{\sigma} \beta_{\sigma}' (\beta_{\sigma} + \mathcal{N})^{-1}
\]
\[ B_{\sigma} \xi' = B_{\sigma} \xi'^{T} - B_{\sigma} \xi'^{T} (N + B_{Q})^{-1} \xi' B_{\sigma} \xi'^{T} \]

and multiplying on the right by \( \xi' (\xi'^{T} \xi')^{-1} \):

\[ B_{\sigma} = B_{\sigma} - B_{\sigma} \xi'^{T} (N + B_{Q})^{-1} \xi' B_{\sigma} . \]  
(15.2)

If we define

\[ A = B_{\sigma} \xi'^{T} \]

\[ \Omega = (N + B_{Q})^{-1} \]  
(17.2)

we can write the solution:

\[ \tilde{y}_{\sigma} - \tilde{y}_{\sigma}^{ex} = A \Omega (\tilde{y}_{Q}^{ex} - \tilde{y}_{Q}) \]  
(18.2)

\[ B_{\sigma} = B_{\sigma} - A \Omega A^{T} \]  
(19.2)

In order to show the correspondence of this method with that using the Lagrange multipliers derived in the first part of the present paper, we will express matrices \( B_{\sigma} \), \( \xi' \) and vector \( y \) by their partitioned elements, so that:

\[ B_{y} = \begin{bmatrix} \alpha_{Q} & 0 \\ 0 & B_{\sigma} \end{bmatrix} \]

\[ \xi_{y} = \begin{bmatrix} \xi^{Q} \\ \xi' \end{bmatrix} \]

\[ y = \begin{bmatrix} \tilde{y}_{Q} \\ \tilde{y}_{\sigma} \end{bmatrix} \]  
(20.2)

\[ \xi = \begin{bmatrix} -\mathcal{U} & \xi' \end{bmatrix} \]

\[ y = \begin{bmatrix} \tilde{y}_{Q} \\ \tilde{y}_{\sigma} \end{bmatrix} \]  
(21.2)

where \( \tilde{y}_{Q} \) and \( \tilde{y}_{\sigma} \) are vectors representing the values given by Eqns. (3/1) and (4/1) respectively, while \( \xi' \) is defined in Eqn. (41.5/1). Eqns. (33/1), (35/1) and (42/1) may then be shown to become:

\[ \xi' = \xi \xi'^{T} = (N + B_{Q}) \]

\[ \begin{bmatrix} \tilde{y}_{Q} \\ \tilde{y}_{\sigma} \end{bmatrix} = \begin{bmatrix} \tilde{y}_{Q}^{ex} \\ \tilde{y}_{\sigma}^{ex} \end{bmatrix} - \begin{bmatrix} \alpha_{Q} \\ B_{\sigma} \xi' \end{bmatrix} \xi^{-1} (\xi \xi'^{T} - \tilde{y}_{Q}^{ex}) \]

\[ \begin{bmatrix} \alpha_{Q} & 0 \\ 0 & B_{\sigma} \end{bmatrix} \xi'^{-1} (N + B_{Q}) \xi^{-1} \xi' B_{\sigma} \xi'^{-1} \]

\[ \begin{bmatrix} \alpha_{Q} & 0 \\ 0 & B_{\sigma} \xi'^{-1} B_{Q} \end{bmatrix} \]

\[ \begin{bmatrix} \alpha_{Q} & 0 \\ 0 & B_{\sigma} \xi'^{-1} B_{Q} \end{bmatrix} \xi'^{-1} \xi' B_{\sigma} \xi'^{-1} \]

(25.2)
The second row of Eqn.(24.2) results equivalent to Eqn.(18.2), while that obtainable from the lower right terms of Eqn.(25.2), i.e.:

$$\rho_y = \rho_y - \rho_y \tilde{f} \tilde{f}^{-1} \tilde{f} \rho_y \quad , \quad (26.2)$$

corresponds exactly to Eqn.(19.2). The first row of Eqn.(24.2) allows to determinate estimates $\tilde{y}_Q$ of integral data corrections, while the other matrix elements in Eqn.(25.2) refer to their covariance (or dispersion) matrix and to correlations with cross section data due to the adjustment procedure.

In disagreement from what suggested by Dragt's comment, in the case matrix $\rho_y$ results diagonal, it does not seem justified to make adjustments for each experiment "one by one, taking care of proper adjustment of $\rho_y$ in each step". Although this procedure might seem reasonable in Bayesian terms of successive sequences of prior/posterior distributions, however it is seen to result correct in the only case where the orthogonal relationship

$$\tilde{f} \rho_y \tilde{f} = D \quad , \quad (27.2)$$

where $D$ represents a diagonal matrix, is satisfied, i.e. in the case where matrix $\tilde{W}$ results diagonal, which does not seem to meet the generality of situations of interest. The "one by one" procedure could be made iterative, but this appears to be a difficult method of solving linear problems involving matrix inversion.

Also the "integral approach" the author proposes as alternative to that, called "differential", previously derived, seems a repetition of the latter, written in equivalent terms. The author proposes to express a given integral parameter of the form

$$Y = \mathcal{L} \tilde{y}_y \quad , \quad (28.2)$$

where $\mathcal{L}$ represents a linear matrix operator, i.e. resulting linearly dependent on the cross sections (which applies properly to perturbations), in terms linear with the "prior" knowledge $\tilde{y}_y^{ex}$ and integral experiments $\tilde{y}_y^{ex}$ avoiding the use of the adjusted values $\tilde{y}_y$. Since this vector, as may be seen from Eqn.(35/1), results a linear function of $\tilde{y}_y^{ex}$, say $\mathcal{L} \tilde{y}_y^{ex}$, where vector $\tilde{y}_y^{ex}$ comprehends $\tilde{y}_y^{ex}$ and $\tilde{y}_y^{ex}$, this proposition results obvious from the formulations already derived, as shown clearly by the following relationship:

$$[\text{Best unbiased estimator of } Y] = \mathcal{L} \tilde{y}_y$$

$$= \mathcal{L} \mathcal{L} \tilde{y}_y^{ex}$$

$$= \mathcal{L} \tilde{y}_y^{ex} \quad (29.2)$$

where by best estimator it is intended that having minimal variance, and where $\tilde{y}_y$ is not to be considered formed.
by auxiliary variables, i.e. "adjusted data having a completely different meaning here", as the author states, but having just the meaning of estimators, which is pertinent to them from the statistical method adopted.


The method of correlation proposed by Rowlands and MacDougall follows the statistical methods described by Linnik [5]. There appear a few points, however, which deserve some comment. The first one concerns the way by which systematic errors of integral parameters are accounted for. The authors include systematic errors in the equation in exactly the same way as the cross sections. More precisely, a product

$$ \mathcal{U} h $$

where $h$ is a vector of $Y$ components representing the systematic errors considered and $\mathcal{U}$ is a $J$ row matrix (J=number of integral experiments) with coefficients $t_{ij}$ either unity or zero depending on whether or not the systematic error $h_j$ applies to the integral experiment $j$, is included systematically in the product $\mathcal{U} h$ in the fitting procedure, i.e. $\mathcal{U}$ appears to have $Y$ more columns (and similarly matrix $\mathcal{N}$ defined by Eqn.(A1.5/I)) while $h$ includes also vector $h$. This amounts to consider new $Q_j^{cal} / Q_j^{obs}$ fractional integral values which are biased with respect to the old ones $Q_j^{cal} / Q_j^{cal}$ by $\sum_y t_{ij} h_y$, i.e.
\[ \frac{q_{j}}{q_{\text{cal}}} = q_{j}^{\text{cal}} + \sum_{y=1}^{Y} \mathbf{t}_{ij} y_{y}^{h} . \]  

(1.3)

If we define also the vector

\[ y^{x} = \begin{pmatrix} y_{Q} \\ y_{F} \\ h \end{pmatrix} = \begin{pmatrix} (y_{Q} + \tilde{y}_{B}) \\ y_{F} \\ h \end{pmatrix} \]  

(2.3)

(where \( y_{Q} \) and \( y_{F} \) represent integral and differential unbiased parameters), the matrix

\[ \mathbf{S}^{x} = \begin{pmatrix} S_{Q} \\ S_{F} \\ h \end{pmatrix} \]  

(3.3)

and replace \( y^{ex} \) by vector

\[ y^{xex} = \begin{pmatrix} y_{Q}^{ex} \\ y_{F}^{ex} \\ h^{ex} \end{pmatrix} \]  

(4.3)

where vector \( h^{ex} \) represent assumed values, all the formulas derived in the text still hold formally once an error covariance matrix \( \mathbf{S}_{h} \) is introduced (correlations between systematic errors are also allowed for). The authors do not comment on this point which appears fundamental. In fact, in presence of systematic errors, a number of possibilities may be encountered:

a) The variances \( D(h_{y}) \) approach infinity (this corresponding to the case where any set of values \( h_{y}^{ex} \) may be assumed), i.e. we have a flat distribution, which is likely to reflect the majority of actual situations when systematic errors are present. In this case the terms at the left hand side of Eqn.(17/1) including the variables \( h_{y} \) have zero coefficients, which corresponds to their having no influence on the minimization of the quadratic term at the exponent of the likelihood function. This implies that the solution \( \hat{y} \) which depends on vector \( y^{xex} \), as shown by Eqn.(35/1), becomes arbitrary. So, no way of dealing with systematic errors is possible in this case, since there is no information at all on their value and distribution.

b) The variances \( D(h_{y}) \) corresponding to the diagonal terms of matrix \( \mathbf{S}_{h} \) approach zero. In this case the adjustment may not be performed for the presence of singularities. On the other hand this implies, as in the case of a cross section (ideally) perfectly known, a precise knowledge of parameters which may be very well included in the vector \( s_{0} \) which appears in Eqn.(13/1). More precisely, in the case of a systematic error, this leads to correcting the biased values \( q_{j}^{y}/q_{\text{cal}}^{y} \) to reduce them to the former unbiased \( q_{j}^{y}/q_{\text{cal}}^{y} \) values.

c) Matrix \( \mathbf{S}_{h} \) has finite diagonal elements and values \( h_{y}^{ex} \)
are assessed. The solutions derived in the text are applicable to this case with vector $y$ substituted by $y^\wedge$ given by Eqn.(2.3), matrix $\mathbf{S}$ by $\mathbf{S}^\wedge$ given by Eqn.(3.3) and correlation matrix $\mathbf{R}_y$ by matrix

$$
\mathbf{R}_{y+h} = \begin{bmatrix}
\mathbf{R}_y & 0 \\
0 & \mathbf{R}_h
\end{bmatrix}.
$$

(5.3)

If we substitute these expressions in the final formulation (35/1) we obtain:

$$
\begin{bmatrix}
\hat{y}_Q \\
\hat{y}_E \\
\hat{h}
\end{bmatrix} = \begin{bmatrix}
\mathbf{y}_Q \\
\mathbf{y}_E \\
\mathbf{h}
\end{bmatrix} - \begin{bmatrix}
\mathbf{B}_y & 0 \\
0 & \mathbf{B}_h
\end{bmatrix} \left( \begin{bmatrix}
\mathbf{S}_y & 0 \\
0 & \mathbf{S}_h
\end{bmatrix} \begin{bmatrix}
\mathbf{B}_y & 0 \\
0 & \mathbf{B}_h
\end{bmatrix} \right)^{-1} \begin{bmatrix}
\mathbf{y}_Q \\
\mathbf{y}_E \\
\mathbf{h}
\end{bmatrix}.
$$

(6.3)

If we split dispersion matrix $\mathbf{R}_y$ and matrix $\mathbf{S}$ in their component parts, relative to the integral and differential parameters represented by vectors $\mathbf{y}_Q$ and $\mathbf{y}_E$ [as done previously with Eqns.(20.2) and (21.2)] and recall that the unbiased vector $\tilde{y}$ along with this procedure is given by the expression:

$$
\tilde{y} = \begin{bmatrix}
\tilde{y}_Q \\
\tilde{y}_E \\
\tilde{h}
\end{bmatrix} = \begin{bmatrix}
\mathbf{y}_Q \\
\mathbf{y}_E \\
\mathbf{h}
\end{bmatrix} - \begin{bmatrix}
\mathbf{B}_Q & 0 \\
0 & \mathbf{B}_h
\end{bmatrix} \left( \begin{bmatrix}
\mathbf{S}_Q & 0 \\
0 & \mathbf{S}_h
\end{bmatrix} \begin{bmatrix}
\mathbf{B}_Q & 0 \\
0 & \mathbf{B}_h
\end{bmatrix} \right)^{-1} \begin{bmatrix}
\mathbf{y}_Q \\
\mathbf{y}_E \\
\mathbf{h}
\end{bmatrix}.
$$

(7.3)

It is simple to see that the procedure suggested by Rowlands et al. for allowing for systematic errors results equivalent to correcting the biased experimental measurements in order to obtaining the unbiased ones.
\[
\begin{align*}
\varepsilon_{\text{ex}} &= \varepsilon_{\text{ex}} - \varepsilon_{\text{ex}}^2, \\
\beta_Q^* &= \beta_Q^* + \gamma \beta_h \gamma^T, \\
\text{and then following the standard correlation procedure.}
\end{align*}
\]

We notice also that the adoption of vector \( \gamma^2 \) and dispersion matrix \( \beta_Q^* \) represents a standard procedure in the course of reduction of experimental values. The simple sum of quadratic terms pertinent to the two matrices at the right hand side of Eqn.(9.3) reflects the fact that the systematic errors which have been introduced in the fitting procedure have been assumed independent from those relative to matrix \( \beta_Q^* \). Further, assuming a null vector \( \gamma^2 \), as the authors have done in their adjustments, represents an operation which results equivalent to the operation of increasing the variances (and covariances) of matrix \( \beta_Q^* \) relative to the experimental integral data. This operation, although beneficial to the discrepancies between the adjusted values and the experimental data, is by far different from accounting for systematic errors. In case there was a suspect of consistent (not systematic) errors in some experimental integral data, it would have been much simpler to properly assess a new dispersion matrix according to Eqn.(9.3).

d) There might be some suspect that a curve shape (for instance related to nuclear temperature \( T \) to be adopted for determining the fission neutron spectrum) is different from the one currently used. How can we incorporate this information into the correlation procedure? The answer to this problem is given by Bayes' theorem, which applies to those cases in which (prior) information pertinent to not random parameters has to be used (but it may be successfully extended also to cases in which prior information on variables is considered, as seen with Dragt's method). It should be emphasized that the theorem is correct once the idea of probability distributions for parameters has been accepted. According to Bayes' theorem, the following relation holds (see discussion on Dragt's method at previous section):

\[
P(T|\gamma^2) = aP(T)L(T|\gamma^2) \tag{10.3}
\]

where \( a \) is a constant, \( P(T|\gamma^2) \) represents the "posterior" (continuous or discrete) distribution for \( T \), \( P(T) \) the "prior" (continuous or discrete) distribution, \( L(T|\gamma^2) \) the likelihood function for parameter \( T \). If, for instance, the value of \( T \) is assumed to be with 90\% probability \( T_1 \) and with 10\% probability \( T_2 \), we obtain the posterior (two value) distribution:

\[
\begin{align*}
P(T_1|\gamma^2) &= 0.9aP(T_1)L(T_1|\gamma^2) \\
P(T_2|\gamma^2) &= 0.1aP(T_2)L(T_2|\gamma^2)
\end{align*}
\]

\[
\begin{align*}
P(T_1|\gamma^2) &= 0.9aP(T_1) = 0.9a \exp \left[ -\frac{(\gamma^2 - (1))}{\beta_1} \right] \\
P(T_2|\gamma^2) &= 0.1aP(T_2) = 0.1a \exp \left[ -\frac{(\gamma^2 - (2))}{\beta_2} \right]
\end{align*}
\]
where \( \hat{y}^{(1)} \) and \( \hat{y}^{(2)} \) represent estimators pertinent to using temperatures \( T_1 \) and \( T_2 \) respectively in the fitting procedure. From this distribution an estimator \( T \) (\( T_1 \) or \( T_2 \)) may be assessed. In case no information is available on the "prior" \( P(T) \) distribution, according to the (much controverted but inevitable in these cases) Bayes' postulate, one would set the "prior" distribution \( P(T) = \text{constant} \). Then, for the case considered it would result: \( P(T_1) = P(T_2) = 0.5 \). In case the prior distribution of \( T \) results flat, for instance

\[
P(T_A < T < T_B) = \text{constant} \quad (12.3)
\]

a few points (for instance three values \( T_1, T_2, T_3 \)) might be chosen and corresponding values of the likelihood function \( L(T|\hat{y}_{\text{ex}}) \) determined. The fitted curve \( L(T|\hat{y}_{\text{ex}}) \) vs. \( T \) would allow to obtain the estimator \( T \) (corresponding to its maximum value) and its distribution.

The second comment is related to the previous one and refers to the quantity \( \bar{R} \) defined by Eqn.(43A). Following the statistical methods, this quantity should be used to test the significance of the results, as has been described in the first part of this work. In cases where the value \( \bar{R} \) is too high, for instance significant at the 5% level of significance, there are serious doubts that some systematic errors are present, either due to experimental inaccuracies, or to deficiencies of the theoretical methods adopted.

and in these directions improvements should be envisaged(\( \star \)). According to this philosophy, it appears unjustified the increase by a factor of 1.2 performed by the authors on all the standard deviations of the cross sections to reduce the value 1.4 which resulted for the \( R \) quantity since this resulted out of the assumed level of significance.

For what concerns finally the confidence intervals, it is true that they may be obtained using the Student's distribution, with \( q \) degrees of freedom,

\[
t_q = \frac{\hat{y}_i - \sigma_i}{\sqrt{[C^{-1}]_{ii}}} \quad (13.3)
\]

[where matrix \( C \) is given by Eqn.(A1.12/1)], which leads to the interval, for the \( i \)-th cross section:

\[
I(i) = [(\hat{y}_i - y)(C^{-1})_{ii}^{1/2} \bar{R}/q],(\hat{y}_i + y)(C^{-1})_{ii}^{1/2} \bar{R}/q
\]

(\( \star \)) Among possible inadequacies of the mathematical model are here included for generality on one hand the approximations made on the theory in the integral data calculations and on the other hand those consequent on not taking proper account, in the construction of the dispersion matrix, of existing (and known) correlations among the nuclear data (for instance due to the theoretical model assumed or to the derivation of cross section experimental values from ratio measurements). These deficiencies sum up in more or less serious lack of exploitation of the existing information.
where $\gamma$ is a number such that

$$\Pr(t_q \leq \gamma) = p,$$

$p$ representing the confidence coefficient (for instance, 95%) chosen. In case the number of integral experiments $q$ is very high, it is also true that the Student's distribution approaches a normal one, but with zero mean and unit variance. Instead, the distributions of the adjusted quantities represented by vector $\tilde{y}$ result normal with any number $q$ of integral experiments, once the normality has been assumed for $y^{\text{ex}}$.

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4. - MITANI AND KUROI'S METHOD [6]

The approach followed by Mitani and Kuroi follows basically and develops the statistical lines proposed by Rowlands and MacDougall and results consistent, in particular, with Linnik's book formulations. In the first part of their paper they propose a method for introducing into the dispersion matrix $\mathbf{\Sigma}_q$ the covariant (off-diagonal) terms related to two types of correlations between the group cross sections: the first one energy-wise in which given nuclear parameters $q_1$ consistent with the nuclear model adopted are introduced, the second one in which cross section measurements relative to standard ones are considered. The first type of correlation is based on the assumption that a given cross section function

$$\sigma(E) = \sigma(E|q_1, q_2, \ldots)$$

(1.4)

is fitted with measured values $\sigma^{\text{ex}}(E_1)$ of the microscopic
cross sections so that estimated values \( \hat{q}_m \) are obtained and, consequently, the semiempirical curve

\[
\hat{\lambda}(E) = (E|\hat{q}_1, \hat{q}_2, \ldots) \tag{2.4}
\]

Together with the dispersion matrix \( \mathcal{B}_q \) which comprehends the (diagonal) variance terms \( \Delta \hat{q}_m^2 \) and the (off-diagonal) covariant terms \( \hat{r}_{mn} \). By group averaging it may be obtained:

\[
\hat{\lambda}_i = \pi_i(\hat{q}_1, \hat{q}_2, \ldots) . \tag{3.4(m)}
\]

If the linear expansion

\[
\hat{\lambda}_i - \pi_i = \sum_m \left( \frac{\partial \pi_i}{\partial q_m} \right) (\hat{q}_m - q_m) \tag{4.4}
\]

is adopted, the following correlation coefficient between group cross sections \( \hat{\lambda}_i \) and \( \hat{\lambda}_j \) may be evaluated:

\[
\gamma_{ij} = E[(\hat{\lambda}_i - \pi_i)(\hat{\lambda}_j - \pi_j)] = \frac{1}{E(o_1 - \pi_i)(o_j - \pi_j)} (A_1^2 - (\pi_i)(\pi_j)) \tag{5.4}
\]

(\( ^{(m)} \)) Not necessarily adopting linear (or linearized) fundamental equations, which is feasible only when corrected-to-first-order estimates \( q_{om} \) of parameters \( q_m \) are available.

(\( ^{(am)} \)) \( \hat{q}_1 \) here corresponds to \( \sigma_1^{ex} \) which appears elsewhere in the text.

With the second type of cross section correlation, if \( \sigma_1^{ex} \) and \( r_{ix}^{ex} \) represent the experimental values of the absolute measurement of the cross section \( \sigma_x \), at a given energy, and the measurement of the ratio

\[
r_{ix} = \frac{\sigma_i}{\sigma_x} . \tag{6.4}
\]

where \( \sigma_i \) represents a given cross section at energy \( E_i \), the variance of \( \sigma_i^{ex} \) results

\[
(\Delta \sigma_i^{ex})^2 = (r_{ix}^{ex})^2(\Delta \sigma_x^{ex})^2 + (\sigma_i^{ex})^2(\Delta r_{ix}^{ex})^2 \tag{7.4}
\]

While, since \( \sigma_i^{ex} \) is independent from \( r_{ix}^{ex} \), the following correlation term is obtained:

\[
S_{ix} = E[(o_1^{ex} - \sigma_i)(o_x^{ex} - \sigma_x)]
= r_{ix}^{ex}(\sigma_i^{ex} - \sigma_x)^2 = r_{ix}^{ex}(\Delta \sigma_x^{ex})^2 , \tag{8.4}
\]

(\( \Delta \sigma_x^{ex} \)) representing the variance of the absolute measurement \( \sigma_x^{ex} \).

A further step which might be suggested with respect to the first of the two types of cross section correlation described above is represented by the adoption of nuclear parameters \( q_m \),
as data to be adjusted directly by means of the integral information. Once the sensitivity coefficients \( \frac{\partial \sigma_t}{\partial q_0} \) have been evaluated, the substitution of \( \sigma_t^{\text{ex}} - q \) with expression (4.4), which may be synthetically written

\[
\hat{\sigma} - \hat{q} = \mathcal{L} (\hat{\sigma} - \hat{q}) ,
\]

in the formulations pertinent to the cross section adjustment leads directly to analogous formulations pertinent to the nuclear parameters \( q_m \). Matrix \( \mathcal{J}' \) as defined by Eqn. (A1.5/1), pertinent to the reduction by elements, would be replaced by matrix \( \mathcal{J}'\mathcal{L} \) (an analogous substitution taking place if the method using the Lagrange multipliers is considered) while the dispersion matrix \( \mathcal{B}_v \) would be replaced by matrix \( \mathcal{B}_q \). Besides, this method would have the advantage of adjusting basic nuclear parameters which can be used directly for different cross section group collapsing.

In relation with the formulations proposed above, concerning the correlations existing among the cross sections, there is no doubt they would improve greatly the cross section adjustment since a more adequate dispersion matrix \( \mathcal{B}_v \) is adopted. But it seems also true that this appears a formidable task to be done, involving a close analysis of all the theory, measurement and evaluation work so far performed.

The authors tackle also the painful problem of the presence of systematic errors within the differential data (1). These errors are assumed to affect only the absolute cross section measurements while the measurements of ratios of cross sections at different energy levels are supposed normally distributed. Therefore, the absolute measurement of cross section \( \sigma_x^{\text{ex}}(E_t) \) at energy \( E_t \) and reaction \( x \), may be expressed as the sum of two terms, i.e.

\[
\sigma_x^{\text{ex}}(E_t) = \sigma_x^{N}(E_t) + \sigma_x^{S}(E_t)
\]

where \( \sigma_x^{N}(E_t) \) represents the normally distributed component, while \( \sigma_x^{S}(E_t) \) its systematic error. If \( r_x^{\text{ex}}(E_i/E_t) \) represents the measurement of the ratio of cross sections of reaction \( x \), at energies \( E_i \) and \( E_t \) respectively, we obtain:

\[
\sigma_x^{\text{ex}}(E_i) = r_x^{\text{ex}}(E_i/E_t) [\sigma_x^{N}(E_t) + \sigma_x^{S}(E_t)]
\]

\[
= \sigma_x^{N}(E_i) + r_x^{\text{ex}}(E_i/E_t) \sigma_x^{S}(E_t) .
\]

Since \( r_x^{\text{ex}}(E_i/E_t) \) is not generally given, the authors propose to approximate it, in the group schematization, by the
where \( \sigma_{x,i}^{ex} \) and \( \sigma_{x,t}^{ex} \) correspond to known cross sections of reaction \( x \) of which the latter one is supposed being obtained from an absolute measurement. With this group schematization Eqn. (12.4) becomes:

\[
\sigma_{x,i}^{ex} = \sigma_{x,i}^{I} + r_{x,i}^{ex} \sigma_{x,t}^{S}.
\] (13.4)

In order to derive the estimates in the adjustment procedure it is sufficient to substitute vector

\[
y_{ex} = \begin{bmatrix} y_{Q}^{ex} \\ y_{r}^{ex} \end{bmatrix}
\] (14.4)

with vector

\[
y' = \begin{bmatrix} y_{Q}^{ex} \\ y_{r}^{ex} - \beta \bar{\sigma}^{S} \end{bmatrix}
\] (15.4)

where \( \bar{\sigma}^{S} \) represent the systematic errors to be associated with the absolute measurements while \( \beta \) represents an assigned matrix of \( I \) rows and \( K \) columns (\( K \) being the number of the absolute measurements considered). The rank of matrix \( \beta \) is supposed equal to \( K \). If it would result less than \( K \), the number of the parameters \( \sigma_{k}^{S} \) could be reduced (by eliminating linear dependences) so that the above mentioned condition results satisfied. Looking into the adjustment equations, it results that the number \( K \) of columns (and, therefore, the number of systematic parameters which may be introduced into the fit) is limited not only by the number \( I \) of the group cross sections, which is obvious, but also by the number \( J \) of the integral parameters considered in the correlation. This results from the necessity that the number \((I+K)\) of parameters to be evaluated result less than the number \((I+J)\) of experimental values, i.e. it should be:

\[
(I+K) < (I+J).
\] (16.4)

Once expression (15.4) is introduced into the likelihood function in place of \( y \), the following expression is obtained:

\[
L = a \exp \left\{ \frac{1}{2} \left[ (y_{Q}^{ex} - y_{Q})^{T} \beta^{-1} (y_{Q}^{ex} - y_{Q}) + (y_{r}^{ex} - y_{r})^{T} \beta^{-1} (y_{r}^{ex} - y_{r}) \right] \right\}.
\] (17.4)

where \( a \) represents a constant value. Since

\[
y_{Q} = \beta' y_{r},
\] (18.4)
setting, for simplicity, in Eqn.(15.4) $\sigma_{0,1}=e^x_1$ (so that $y^{ex}_0=0$), substituting in Eqn.(17.4) and deriving with respect to $y'_0$ and $\sigma^S$ one obtains the normal equations:

$$y^T B^{-1}(y^{ex} - y'^TQ) - B^{-1}(y'_Q + R) = 0$$

From the second equation we have

$$\tilde{S} = -(R B^{-1}R^{-1} R B^{-1} y'_Q)$$

where, since $\text{rank}(R)=K$, the matrix in parenthesis may not be singular. Substituting this expression in Eqn.(19.4) gives:

$$\left[ y'^T B^{-1} y' + B^{-1} R (R B^{-1} R^{-1} R B^{-1}) y'_Q \right] = 0$$

$$= y'^T B^{-1} y'$$

which, apart from the third term in the sum in square parenthesis, is formally equivalent to the expression (A1.15/1), i.e. to the solution in the case where no systematic errors are present. This means that the expressions which give $y'_Q$ (and $y'_Q$) and the dispersion matrix $B^S$ result equivalent to those already derived for the normal case provided the dispersion matrix $B^S$ is replaced by the matrix $\sigma^S$.

As a consequence, once matrix $(B^S_y)^{-1}$ has been evaluated, a correlation code already developed for normal cases may be used, provided that inequality (16.4) is respected. On the other hand, it may be also easily verified, by substituting Eqn.(21.4), that in Eqn.(17.4), which gives the likelihood function, the term in square parenthesis at the exponent becomes

$$\left[ y^{ex}_Q - y'_Q \right] T B^{-1} (y^{ex}_Q - y'_Q) + (y^{ex}_Q - y'_Q)^T (B^S_y)^{-1} (y^{ex}_Q - y'_Q)$$

Since the number of degrees of freedom is now given by the difference

$$r = \text{rank}(\tilde{S}^{-1}) - \text{rank}(R)$$

the distribution of the scalar product

$$\tilde{K} = \tilde{Y}^T B^{-1} \tilde{Y}$$

(results having a $\chi^2$ distribution).

Here one point deserves a little comment. In this work (and also elsewhere) the presence of a coefficient $\sigma^2$ is associated with the dispersion matrices and finally estimated by the ratio [recalling expression (43A)].

$$\left( B^S_y \right)^{-1} B^S_y - B^S_y R (R^T B^S_y R)^{-1} R^T B^S_y$$

Since matrix $(B^S_y)^{-1}$ results singular (see Appendix 1), which appears acceptable for the inverse of a dispersion matrix, the reduction by elements (as shown in Appendix 1 of Part 1) should be adopted.
This procedure (followed for generality in many texts of statistics, for instance in Linnik's book) is valid in important cases when many observations (for instance angular measurements in geodesy) are made with the same instrument, the precision ($\sigma^2$) of which is not known. It is natural then to express the variance of an experimental value as given by $\sigma^2$ divided by the number of observations. Then this applies to a set of experimental values (in the above example, corresponding to a set of angular points), it is then convenient to put in evidence the common factor $\sigma^2$ in the dispersion matrix and to estimate it at the end of the fit. But in our case the situation appears quite different, since the dispersion matrix is constructed of (supposedly) known terms and relative to diverse cross section and integral data experiments which by no means may share any factor in common. In this case, introducing such a factor, corresponds to multiply the results by an estimate of unity consistent with the data. As a consequence of this fact, a few formulations relative to the standard deviations of the cross section estimates (corresponding to the diagonal terms of the resulting dispersion matrix) appear here to be redundant.

Ant of this extra term. This same conclusion applies to the formulations and results which appear in Ref.[7], where the method is applied to a specific correlation problem. As a consequence of the removal of this extra term, a few difficulties inherent with the interpretation of the results should presumably disappear. In particular the expression [see Appendix 2]

$$\frac{1}{1+J} \text{Tr} \left[ \beta_y (\beta_y^D)^{-1} \right] = \frac{1}{1+J} ,$$

(28.4)

where $\beta_y^D$ represents a diagonal matrix with the diagonal line equal to that of the dispersion matrix $\beta_y$, should be used merely for calculation checking purposes [5]. To be noticed that the expression at the left hand side of Eqn.(28.4) represents the average value of the estimated variances of the group cross sections and integral parameters, normalized to the experimental ones. As expected, this average value is seen to tend to zero with the number J of integral data tending to infinity.

(*) Under this regard, it should be also noticed that the information inherent within the residual quantity R may be exploited only once: either (if this is the case) for the evaluation of the $\sigma^2$ quantity or for making a $\chi^2$ test.

(**) In particular the standard deviation expression relative to the case in which the presence of systematic errors is accounted for. For the case $\text{rank}(R) \rightarrow \text{rank}(A')$ the ratio (27.4) still represents an estimate of unity since both the denominator and the numerator tend uniformly to zero. The increase of the standard deviations $\Delta y_{\sigma,i}$ consequent in this case on the reduction of the degrees of freedom is connected directly with the matrix structure of the expression which gives it.
and to unity with the number $I$ of group cross sections tending to infinity (in this case the information inherent within a finite number of experimental integral data for the assessment of an infinite number of unknown parameters becomes negligible).

**APPENDIX 1**

Although the singularity of matrix $(G_0)^{-1}$ has not been mathematically proved, however in a number of cases of interest a double inversion check on a program, which has given definite indications in this sense. Besides, it is possible to verify the singularity in a few simple cases which will be shown in the following.

Let us assume the simplest case of one integral parameter and two differential ones, which will be shown in the following.

Let us set $x_1 = x_2 = 1$

The constraints are represented by the equation:

\[
\begin{align*}
x_1^2 + x_2^2 = 1
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x_1}{\partial y_1} = 0, \\
\frac{\partial x_2}{\partial y_2} = 0
\end{align*}
\]

\[
\begin{align*}
x_1 = y_1 + y_2 + y_3
\end{align*}
\]
and unitary dispersion matrices. If for the solution we choose the method of reduction by elements (see Appendix 1 of Part I), defining:

\[
\mathbf{a} = \begin{bmatrix} a_1 \\
                        a_2 \\
\end{bmatrix} = \begin{bmatrix} y_2 \\
                                    y_3 \\
\end{bmatrix}
\]

\[
\mathbf{S}' = \begin{bmatrix} s_{11}' & s_{12}' \\
                       s_{11}' & s_{12}' \\
\end{bmatrix}
\]

\[
\mathbf{P} = \begin{bmatrix} 1 & 0 \\
                      0 & 1 \\
\end{bmatrix}
\]

\[
\mathcal{E}^{-1} = (\mathbf{P}^T \mathbf{S}_y^{-1} \mathbf{P})^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\
                       -1 & 2 \\
\end{bmatrix} = \mathbf{B}_a
\]

the estimate of vector \(\mathbf{a}\) results the following:

\[
\tilde{\mathbf{a}} = \mathcal{E}^{-1} \mathbf{P}^T \mathbf{S}_y^{-1} \mathbf{y}_{ex} = \begin{bmatrix} -1/6 \\
                        -1/6 \\
\end{bmatrix}
\]

Of course, the same solution could have been obtained if we had applied the method of the Lagrange multipliers. If we now allow for the presence of a systematic error in the parameter \(a_{1}^{ex}\) by means of the matrix

\[
\mathbf{R} = \begin{bmatrix} 1 \\
                       0 \\
\end{bmatrix}
\]

by applying Eqn.(23.4) the following inverse dispersion matrix \((\mathbf{R}_a)^{-1}\) is obtained:

\[
(\mathbf{R}_a)^{-1} = \begin{bmatrix} 0 & 0 \\
                                  0 & 1 \\
\end{bmatrix}
\]

which results clearly singular. It is evident that the method of the Lagrange multipliers cannot be applied to this case, since this method implies the inversion of matrix \((\mathbf{R}_a)^{-1}\). On the other hand the singularity of this matrix may be simply justified by the fact that associating a possible systematic error to the experimental value \(y_{1}^{ex}\) results equivalent to associating to it an infinite variance. Consequently, as we shall see, for assessing its value, the whole information contained in the (only one) integral measurement has to be exploited. In fact, adopting the method of reduction by elements and going through the same formulations above, inserting \((\mathbf{R}_a)^{-1}\) in place of \(\mathbf{R}_a^{-1}\), we obtain:

\[
\mathcal{E}^{-1} = \begin{bmatrix} 2 & -1 \\
                       -1 & 1 \\
\end{bmatrix} = \mathbf{B}_a
\]

\[
\tilde{\mathbf{a}} = \begin{bmatrix} -1/2 \\
                      0 \\
\end{bmatrix}
\]
i.e. the whole (systematic) error is associated to $\alpha_1$. This may be verified also by determining the systematic error $\sigma_S$ by means of Eqn.(21.4), which here results in fact equal to 1/2.

It may be seen very quickly that even if matrix $\mathbf{R}$ would have been of the type

$$\mathbf{R} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

with values $\alpha_1$, $\alpha_2$ arbitrary, and always assuming for simplicity unitary dispersion matrices, matrix $(\mathbf{R} \sigma)^{-1}$ remains singular. An analogous conclusion may be obtained directly in the case of three differential parameters and a matrix $\mathbf{R}$ of the type

$$\mathbf{R} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

with $\alpha_1$, $\alpha_2$, $\alpha_3$ arbitrary. If, for instance, we consider the fundamental equations

$$y_1 = a_1 + a_2$$

$$y_2 = a_1 + a_3$$

setting

$$a_1 = a_2 = a_3 = 0$$

$$y_1 = -\frac{1}{2}; \quad y_2 = 0$$

and unitary dispersion matrices, in the case systematic errors are not allowed for, gives the estimates:

$$\mathbf{C}^{-1} = \frac{1}{8} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & 1 \\ -2 & 1 & 5 \end{bmatrix} = \mathbf{R}_a$$

$$\mathbf{a} = \frac{1}{8} \begin{bmatrix} -1 \\ -3/2 \\ 1/2 \end{bmatrix}$$

while, with allowance of a systematic error in the measurement $a_1$, by means of matrix

$$\mathbf{R} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

the following results are obtained:
\[ \mathcal{C}^{-1} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \mathcal{B}_{a} \]

\[ \mathfrak{a} = \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix} \]

\[ \mathcal{S} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

i.e. the adjusted value \( \hat{a}_1 \) consists merely of the systematic error contribution, while in this case, since \( K<J \), also non null normal adjustments \( \hat{a}_2 \) and \( \hat{a}_3 \) are present.

If \( \mathcal{R} \) is chosen to be

\[ \mathcal{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \]

i.e. two systematic errors are assumed, for parameters \( a_1 \) and \( a_2 \) respectively, the results are:

\[ \mathcal{C}^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \mathcal{B}_{a} \]

\[ \mathfrak{g}^s = \begin{bmatrix} 0 \\ -1/2 \\ 0 \end{bmatrix} \]

\[ \mathfrak{g}^s = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} \]

As may be seen, in the dispersion matrix \( \mathcal{B}_{a} \) the variances result increased (since the number of the systematic errors is increased) while, since now \( K<J \), the adjustments consist merely of systematic error contributions.
APPENDIX 2

The matrix which appears at the left hand side of Eqn. (28.4)

\[ B_y (B_y^D)^{-1} \quad (A2.1) \]

recalling the definition (42/1) of \( \beta_y \), may be written:

\[(\Psi - \Xi) B_y^* \quad (A2.2)\]

where \( \Xi \) is defined by the Eqn. (39/I) and where:

\[ B_y^* = B_y (B_y^D)^{-1} = \begin{vmatrix}
1 & b_{1,2}^* & \cdots & b_{1,n}^* \\
b_{1,2}^* & 1 & \cdots & b_{2,n}^* \\
\vdots & \vdots & \ddots & \vdots \\
b_{1,n}^* & b_{2,n}^* & \cdots & 1
\end{vmatrix} \quad (A2.3)\]

\( b_{i,j}^* \) (i\#j) representing given off-central matrix elements. As
shown in Appendix 2 of Part 1, an orthogonal matrix \( \mathcal{F} \) may be
found such that (recalling that \( \mathcal{F}^T = \mathcal{F}^{-1} \)):

\[
\mathcal{F}^T \Theta_y^{\frac{1}{2}} \Theta_y^{\frac{1}{2}} \mathcal{F} = \mathcal{D}
\]  \( \text{(A2.4)} \)

where \( \mathcal{D} \) is a diagonal matrix with elements \( d_i^2 = 1 \) or 0 and
such that

\[
\sum_{i=1}^{N} d_i^2 = J
\]  \( \text{(A2.5)} \)

Setting

\[
2 = \Theta_y^{\frac{1}{2}} \mathcal{F}
\]  \( \text{(A2.6)} \)

Eqn. (A1.3) may therefore be written

\[
2 2^{-1} (\mathcal{U} - \mathcal{I}) 2 2^{-1} \Theta_y^* = 2 \Theta_y^* 2^{-1} \Theta_y^* \]  \( \text{(A2.7)} \)

where \( \Theta_y^* \) represents a diagonal matrix with elements \( d_i^2 = (1-d_i) \).

Since the trace of a product of any number of symmetric matrices results independent from their order and recalling
expression (iii) [see Note at the end of Appendix], the following expression results:

\[
\text{Tr}(2 \Theta_y^* 2^{-1} \Theta_y^*) = \text{Tr}(2 \Theta_y^* \Theta_y^* 2^{-1})
\]  \( \text{(A2.9)} \)

\[
= \text{Tr}(\Theta_y^* \Theta_y^*) = I
\]

as requested.

**NOTE.** - Let us assume first two generic \( \mathbb{N} \times \mathbb{N} \) matrices \( \mathcal{A} \) and
\( \mathcal{B} \). The trace of their product has the form

\[
\text{Tr}(\mathcal{A} \mathcal{B}) = \sum_{i,k=1}^{N} a_{ik} b_{ki}
\]  \( \text{(i)} \)

which is clearly invariant with respect to interchanging the elements of matrices \( \mathcal{A} \) and \( \mathcal{B} \). As a consequence, the well known relationship may be derived

\[
\text{Tr}(\mathcal{A} \mathcal{B}) = \text{Tr}(\mathcal{B} \mathcal{A})
\]  \( \text{(ii)} \)

which (assuming \( \mathcal{B} \) is non-singular) results equivalent to

\[
\text{Tr}(\mathcal{B} \mathcal{A} \mathcal{B}^{-1}) = \text{Tr} \mathcal{A}
\]  \( \text{(iii)} \)

Considering then three generic \( \mathbb{N} \times \mathbb{N} \) matrices \( \mathcal{A} \), \( \mathcal{B} \)
and \( \mathcal{C} \) and two generic indexes \( i,j \), the sum of all the terms of the product matrix \( (\mathcal{A} \mathcal{B} \mathcal{C}) \) containing
the two symmetrical elements \( b_{ij} \) and \( b_{ji} \) results given by the expression

\[
(b_{ij} \sum_{k=1}^{N} a_{ki} c_{jk} + b_{ji} \sum_{k=1}^{N} a_{kj} c_{ik})
\]  \( \text{(iv)} \)

which clearly appears invariant with respect to interchanging the elements of matrices \( \mathcal{A} \) and \( \mathcal{C} \),
in the only case of matrices \( \mathcal{A} \), \( \mathcal{B} \) and \( \mathcal{C} \) symmetric.
In this case it reduces to the expression:

\[
b_{ij} \sum_{k=1}^{N} (a_{ik} c_{jk} + a_{kj} c_{ik})
\]  \( \text{(v)} \)

We may therefore conclude that in this case
As a corollary, combining properties (iii) and (vi), one may deduce that the trace of the product of any number of symmetric matrices results independent from their order.

\[
\text{Tr}(A \otimes C) = \text{Tr}(C \otimes A).
\]

REFERENCES


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