Letter to the Editor

On the HGPT perturbation method for burn-up evolution analysis

In previous works (Gandini, 1981, 1987, 1997) a perturbation method was described for analysis of functionals of neutron and nuclide densities evolving during the core life of a nuclear reactor. The method was founded on the heuristically-based Generalized Perturbation Theory (GPT) and thereafter defined HGPT for distinguishing it from other methods, in particular those adopting variational techniques, as the method developed by Williams (1979) for fuel depletion analysis.

The motivation of this letter is to emphasize a major difference between HGPT and Williams’ methods, difference consisting in the choice of the control parameter. In Williams’ method the neutron flux is factorized in the product of a normalized space and energy function as the boron concentration in the coolant, or the axial position of a control rod, or (fictitiously) the space between HGPT and Williams’ methods, difference consisting in the reactor core life, or the time interval between two shuffling operations.

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1. The HGPT method

The physical behaviour of a nuclear reactor during its life cycle is described by the time evolution of the multigroup neutron density vector n(r, t) and of the nuclide density vector c(t). An intensive control parameter r(t) is introduced for maintaining constant the power (W) assigned. In the following, for simplicity, W shall be intended to represent the total fission rate rather than the power. Functions n, c and r at quasi-static conditions (dt = 0) satisfy equations

\[ \begin{align*}
\mathbf{m}(\mathbf{n}, \mathbf{c}, \rho|\mathbf{p}) &= -\frac{\partial}{\partial t} + B(\mathbf{c}, \rho|\mathbf{p})\mathbf{n} + \delta(t - t_0)\mathbf{n}_s = 0 \\
\mathbf{m}(\mathbf{n}, \mathbf{c}, \rho|\mathbf{p}) &= -\frac{\partial}{\partial t} + B(\mathbf{c}|\mathbf{p})\mathbf{n} + \delta(t - t_0)\mathbf{n}_s + \rho_s(\mathbf{p}) = 0
\end{align*} \]

(1)

where \( B \) is the diffusion or transport operator (dependent on c and, generally, on \( \rho \)), \( E \) the nuclide evolution matrix (dependent on \( \mathbf{n} \)), \( \mathbf{p} \) the system parameters vector, \( \mathbf{n}_s \) and \( \mathbf{s}_r \) are source terms, while \( S \) is a matrix with elements given by microscopic fission cross sections \( \sigma_{\text{ff}} \) \( [m = 1...M] \) (number of nuclides), \( g = 1...G \) (number of groups). The terms with Dirac deltas correspond to initial conditions. The source term \( s_r \) in Eq. (2) is generally given by a sum of delta functions defined at specific times to account for fuel load and shuffling operations.

In Eq. (1) for the sub-critical case, control over the source has been chosen. If the control were defined within operator \( B \), the methodology would result similar to that for the quasi-critical reactor.

A general form of a (linear) response \( Q \) can be written in the form.

\[ Q = \int_{t_0}^{t} \langle <\mathbf{h}_{n_s}^T\mathbf{n}_{\text{sys}} + <\mathbf{h}_{c}^T\mathbf{c}_{\text{sys}} + \mathbf{h}_{\rho}^T\rho \rangle \rangle \]

(4)

where \( \mathbf{h}_{n_s}^T, \mathbf{h}_{c}^T \) and \( \mathbf{h}_{\rho}^T \) are given quantities, depending on the functional considered.

From the linear expansion around a given solution of Eqs. (1)–(3) we can first obtain the linear equations governing the derivative functions and then, by reversing the governing operators, the adjoint equations governing the adjoint.

The system of equations governing the derivative functions results, recalling the coordinates complementation rule (Gandini, 1987):

\[ \begin{align*}
\mathbf{m}(\mathbf{n}, \mathbf{c}, \rho|\mathbf{p}) &= -\frac{\partial}{\partial t} + B(\mathbf{c}, \rho|\mathbf{p})\mathbf{n} + \delta(t - t_0)\mathbf{n}_s = 0 \\
\mathbf{m}(\mathbf{n}, \mathbf{c}, \rho|\mathbf{p}) &= -\frac{\partial}{\partial t} + B(\mathbf{c}|\mathbf{p})\mathbf{n} + \delta(t - t_0)\mathbf{n}_s + \rho_s(\mathbf{p}) = 0
\end{align*} \]

\[ Q = \int_{t_0}^{t} \langle <\mathbf{h}_{n_s}^T\mathbf{n}_{\text{sys}} + <\mathbf{h}_{c}^T\mathbf{c}_{\text{sys}} + \mathbf{h}_{\rho}^T\rho \rangle \rangle \]

In case that one or more variables do not depend on one or more space-time coordinates [for example the control parameter \( r(t) \) which depends only on time] by this rule the methodology assumes that these variables correspond to integral values (for example, volumetric averages) of fictitious quantities dependent on all the coordinates of the phase space. They are then replaced with expressions with averaging operators of the type \( \langle \rightarrow \rangle \) or simply \( < \rightarrow > \). In the example considered we would then have \( < r(r, t) >_{\text{sys}} \) (\( \mathbf{n} \)). In the final results, only the space integrated importance’s associated with these variables will be of interest.

The coordinates complementation rule is required in the operator reversal procedure for obtaining the correct operators governing the importance functions.
the nuclide importance, results:

\[
\frac{d}{dt} \left( \frac{\partial \rho}{\partial \Omega} \right)_{m} \frac{\partial \rho}{\partial \Omega} - \frac{\partial \rho}{\partial \Omega} \frac{\partial \rho}{\partial \Omega} m_{n, j} = 0
\]  

(5)

where \( \Omega_{n} \) and \( \Omega_{c} \) are coupling operators given by the Frechet derivatives \( \frac{\partial \rho}{\partial \Omega} \) and \( \frac{\partial \rho}{\partial \Omega} \), respectively, and where \( \omega_{p} \) is given by the expression

\[
\omega_{p} = \frac{\partial \rho}{\partial \Omega} \Omega_{n} + (1 - \kappa) s_{n} ~ (\kappa = 1 \text{ for quasi-critical}, \kappa = 0 \text{ for critical systems})
\]  

(6)

By reversing the operators, the system of equations governing the importance functions results:

\[
\left( \begin{array}{c}
\frac{d}{dt} + B \end{array} \right) \Omega_{c} \left( \begin{array}{c}
S' c < > \gamma_{sy} \\
\Omega_{n} \end{array} \right) = \left( \begin{array}{c}
\frac{d}{dt} + E \end{array} \right) S_{n} < > \gamma_{sy} \left( \begin{array}{c}
\mathbf{n} \\\n\mathbf{c} \end{array} \right) + \mathbf{h}_{c}^{*} = 0
\]  

(7)

where \( \omega_{p}, \Omega_{n}, \text{ and } \Omega_{c} \) are the adjoints of operators \( \omega_{p}, \Omega_{n}, \text{ and } \Omega_{c} \), respectively.

Setting \( \rho' = -\frac{\partial \rho}{\partial \Omega} \), the equation corresponding to the first row, relative to the neutron importance, results:

\[
\frac{d}{dt} \mathbf{n} + B' \mathbf{n} + \Omega_{c} \mathbf{c} + S' \mathbf{c} \rho' + \mathbf{h}_{c}^{*} = 0
\]  

(8)

while the equation corresponding to the second row, relative to the nuclide importance, results:

\[
\frac{d}{dt} \mathbf{c} + E' \mathbf{c} + \Omega_{n} \mathbf{n} + S \mathbf{c} \rho' + \mathbf{h}_{c}^{*} = 0
\]  

(9)

The equation corresponding to the third row, relevant to the importance associated with the control, results, recalling Eq. (6),

\[
\kappa < \mathbf{n} > \frac{\partial \rho}{\partial \Omega} \mathbf{n} > \gamma_{sy} + (1 - \kappa) < \mathbf{n} > \gamma_{sy} + \mathbf{h}_{p}^{*} = 0.
\]  

(10)

If \( \mathbf{h}_{p}^{*} = 0 \), this equation indicates that in quasi-critical systems the neutron importance function in the phase space is orthogonal to the neutron density distribution, while in sub-critical systems it is orthogonal to the extraneous neutron source distribution if this source is associated with the control.

Assuming \( \mathbf{h}_{p}^{*} = 0 \), multiplying by \( \mathbf{n} \) to the left Eq. (8) governing \( \mathbf{n} \) and integrating we obtain:

\[
\rho' = -\frac{\mathbf{n} \Omega_{c} \mathbf{c} > \gamma_{sy}}{\mathbf{W}}.
\]  

(11)

The sensitivity coefficient \( \frac{\partial \mathbf{n}}{\partial \rho} \) with respect to a system parameter \( \rho \) can be written in the form:

\[
\frac{d}{dt} \mathbf{n} + \int_{t_{0}}^{t} \left( \frac{\partial \mathbf{n}}{\partial \rho} \right)_{m} \mathbf{m}_{n, j} + \mathbf{c} \left( \frac{\partial \mathbf{c}}{\partial \rho} \right)_{m} \mathbf{m}_{c, j} = 0
\]  

(12)

In practical cases, for solving Eqs. (8)-(10), a time-step discretization and a number of core macro-zones with average density values of the evolving nuclides are defined, together with a recurrent calculation procedure, starting from the given adjoint sources \( \mathbf{h}_{c}^{*}, \mathbf{h}_{c}^{*}, \text{ and } \mathbf{h}_{p}^{*} \).

2. Williams’ method

With Williams’ method it is assumed that, for a quasi-critical system, the neutron density \( \mathbf{n} \) is normalised so that \( \mathbf{u} \mathbf{n} > \gamma_{sy} = 1 \), \( \mathbf{u} \) being a unit vector. The normalized function \( \mathbf{n} \) remains however a function of time, its space-energy distribution evolving with the burn-up.

Eqs. (1), (2) and (3) in this case, at critical conditions, may be rewritten as

\[
\begin{align*}
\mathbf{m}_{n,o}(\mathbf{n}, \mathbf{c}) &= -\frac{\partial \mathbf{n}}{\partial \mathbf{c}} + B(\mathbf{c} \mathbf{p}) \mathbf{n} + \delta(t - t_{0}) \mathbf{n}_{o} &= 0 \\
\mathbf{m}_{c,o}(\mathbf{n}, \mathbf{c}) &= -\frac{\partial \mathbf{c}}{\partial \mathbf{c}} + E(\mathbf{p} \mathbf{c} \mathbf{c} + \delta(t - t_{0}) \mathbf{c}_{s} + \mathbf{s} = 0
\end{align*}
\]  

(13)

where \( \rho \) plays the role of an intensive, time-dependent control parameter. The product \( \mathbf{p} \mathbf{n} \) corresponds then to the real neutron density.

It may be demonstrated that Williams’ method is equivalent to the HGPT method in case of functionals of neutron or nuclide densities, whereas the HGPT method allows considering as an additional functional subject to analysis the control parameter itself, as illustrated in the elementary example below.

To be reminded that the equivalence between the methods using the GPT theory and those using the variational theory has been shown by Greenspan (1975) and later by Gandini (1987). This equivalence is of course valid both in absence or in presence of the neutron flux factorization.

3. Elementary example

Consider a homogeneous fuel cell in the core of a nuclear system with one fissile nuclide of density \( \mathbf{c} \). During the burn-up the fission rate density \( \mathbf{W} \) is maintained constant by an intensive control function \( \rho \), fictitiously assumed as a coefficient of the neutron fission source.

The functional considered is the control function at end of cycle, i.e., \( \rho(t_{f}) \).

The equations relevant to functions \( \mathbf{n}, \mathbf{c} \) and \( \rho \) result, setting for the fissile nuclide \( \sigma_{r} = \sigma_{r} + \sigma_{r} \) and defining with \( \Sigma \) the macroscopic cross section associated with all other elements,

\[
\begin{align*}
\frac{d}{dt} \mathbf{n} + (\rho \mathbf{c} - \sigma_{r} \mathbf{n}) - \Sigma & = 0 \\
\frac{d}{dt} \mathbf{c} + \delta(t - t_{0}) \mathbf{c}_{s} &= 0 \\
\mathbf{W} - \sigma_{r} \mathbf{c} &= 0
\end{align*}
\]  

(14)

with solutions

\[
\mathbf{n} = \frac{\mathbf{W}}{\sigma_{r}} \mathbf{c} = c_{s} - \sigma_{r} \mathbf{W}(t - t_{0}) \quad \rho = \frac{\sigma_{r} + \Sigma}{\sigma_{r} \mathbf{c}}
\]  

(15)

Deriving \( \rho \) with respect to \( c_{s} \) at time \( t_{f} \) gives directly the sensitivity coefficient:

\[
\frac{d\rho(t_{f})}{dc_{s}} = -\frac{\Sigma}{\sigma_{r} \mathbf{c}^{2}}
\]  

(16)

For this elementary case the system of equations governing the importance functions results:

\[
\begin{pmatrix}
\frac{d}{dt} + (\rho \mathbf{c} - \sigma_{r} \mathbf{n}) - \sigma_{r} \mathbf{c} \sigma_{r} \mathbf{c} \\
(\rho \mathbf{c} - \sigma_{r} \mathbf{n}) \mathbf{n} + (\delta(t - t_{0}) \mathbf{c}_{s}) / \mathbf{c}_{s} \mathbf{n} = 0 \Sigma
\end{pmatrix} = 0
\]  

(17)

The equation corresponding to the third row gives the solution for the neutron importance:

\[
\mathbf{n} = -\frac{1}{\mathbf{n} \sigma_{r} \mathbf{c}} \delta(t - t_{0}).
\]  

(18)

Replacing it in the equation relevant to \( \mathbf{c}^{*} \) allows obtaining the solution for the nuclide importance.
\[ \begin{align*}
C' &= -\frac{\rho(t_F)\nu \sigma_t - \sigma_c}{\nu \sigma_t c(t_F)} = -\frac{\sum}{\nu \sigma_t c^2(t_F)} \\
\end{align*} \] (19)

which results corresponding to a constant value, independent of time.

Along with expression (12), the sensitivity coefficient of the functional considered results

\[ \frac{d\rho(t_F)}{dc_0} = \int_{t_o}^{t_F} \frac{d\rho}{dc_0} = \int_{t_o}^{t_F} \sigma(t - t_o)c'dt = C'(t_o) = -\frac{\sum}{\nu \sigma_t c^2(t_F)} \] (20)

which corresponds exactly to the expression obtained directly, Eq. (16).

References


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