A GENERALIZED PERTURBATION METHOD FOR BI-LINEAR FUNCTIONALS OF THE REAL AND ADJOINT NEUTRON FLUXES

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Abstract—A generalized perturbation method is derived with respect to ratios of bi-linear functionals of the real and adjoint neutron fluxes of critical multiplying systems. Simple linear analysis for optimization and sensitivity studies are then feasible relative to spectrum and space-dependent quantities, such as Doppler and coolant void reactivity effects in fast reactors.

1. INTRODUCTION

Before describing new perturbation methods for nuclear reactor analysis, it seems appropriate to define first the limits of validity of conventional perturbation theory, on which the methods presently in use are based. For easier understanding we will use the concept of 'observable' which represents a measurable quantity in a reactor system. The 'observable' hitherto considered in conventional perturbation theory has been the reactivity worth associated with some change of the nuclear constants introduced into the system under investigation. We will report below the rigorous formulation derived in 1955 by USACHEV (1955). To simplify the nomenclature we will use here a multigroup approximation of the transport equation. Adopting a vector and matrix notation, the real and adjoint neutron fluxes \( \psi(r, \Omega) \) and \( \psi^*(r, \Omega) \) of a critical reactor are given by the equations:

\[
\begin{align*}
\mathbf{P} \psi &= \mathbf{A} \psi + \mathbf{F} \psi = 0 \\
\mathbf{P}^* \psi^* &= \mathbf{A}^* \psi^* + \mathbf{F}^* \psi^* = 0
\end{align*}
\]

where \( \mathbf{P} \) and \( \mathbf{P}^* \) are the real and adjoint multigroup Boltzmann matrix operators, \( \mathbf{F} \) and \( \mathbf{F}^* \) represent the real and adjoint fission source matrices, \( \mathbf{A} \) and \( \mathbf{A}^* \) represent the real and adjoint matrix operators including the diffusion, absorption and down-scattering processes. If we now introduce a perturbation \( \Delta \mathbf{P} \) into the system, the exact reactivity worth as derived by Usachev is given by the equation:

\[
\frac{\Delta K}{K} = \frac{\int \psi^* \Delta \mathbf{P} \psi' \, dr \, d\Omega}{\int \psi^* \mathbf{F} \psi' \, dr \, d\Omega},
\]

where \( \psi' \) and \( \mathbf{F}' \) represent respectively the flux and the fission source matrix of the perturbed system.

Equation (2) is not applicable for practical purposes since the flux \( \psi' \) is not generally known. However, the assumption is usually made that for relatively small perturbations \( \Delta \mathbf{P} \), \( \psi' \) may be replaced by the unperturbed flux \( \psi \), and \( \mathbf{F}' \) is replaced...
by the unperturbed fission source matrix $\mathcal{F}$. The approximate formula which is normally used is then given by the equation:

$$\left( \frac{\Delta K}{K} \right)_{\text{approx}} = \frac{\int \varphi^* \Delta \mathcal{F} \varphi \, dr \, d\Omega}{\int \varphi^* \mathcal{F} \varphi \, dr \, d\Omega}.$$  \hspace{1cm} (2a)

It should be noted that the use of this expression implies an error which is always proportional to the perturbation $\Delta \mathcal{F}$. Although this perturbation may be very small, the resulting relative error is never negligible.

Let us consider now an 'observable' represented by an integral quantity of the unperturbed system, for instance a reaction rate or a reactivity worth, which in this case does not represent a perturbation but is considered as a detectable integral parameter of the reactor. Suppose now that a perturbation $\delta \mathcal{F}$ is permanently introduced into the steady-state system, and is such that the system remains critical. Then the real and adjoint flux shapes change to new steady-state distributions, and consequently the integral quantities referred to will be proportionally perturbed.

On the basis of what has been said above, it seems useful to develop methods which allow one to calculate the effects on given integral quantities due to changes of the real and adjoint neutron fluxes produced by given perturbations of a reactor system.

In general, reactor physics is concerned with integral quantities which are linear with respect to the real and/or adjoint neutron fluxes. Such quantities may be then called linear or bilinear functionals of these fluxes. Linear functionals of the real flux may represent reaction rates, and linear functionals of the adjoint flux may represent neutron source worths. The ratios between two bilinear functionals of the real and adjoint neutron fluxes may represent reactivity worths, reactivity ratios, or prompt neutron lifetimes.

Usachev later developed a method (USACHEV, 1963) by which the effect of the neutron real flux change produced by a given perturbation, on a functional linear with respect to the real flux, may be calculated. The formulation he derives is quite similar to the one used in conventional perturbation theory. Of course Usachev's formulation is applicable also to bilinear functionals of the real and adjoint fluxes, such as reactivity worths and neutron lifetimes, but in this case no allowance is made for the adjoint flux changes produced by the perturbation, which may have an important effect. In the present work this limitation is removed and a more general formulation derived.

2. LINEAR FUNCTIONALS OF THE REAL AND ADJOINT NEUTRON FLUXES

The development of linear perturbation formulations connected with integral detectable processes, stems from the definition of importance function related to an 'observable' defined in space and time. In this respect a clear definition of importance has been given by LEWINS (1960). He considered the general case of a reactor not necessarily in a steady state and then defined a time-dependent importance function. His definition is that "the importance of a neutron or precursor is the probable contribution to an arbitrary detectable process at a selected time $t_r". As a consequence
Bi-linear functionals of the real and adjoint neutron fluxes

he derives the axiom that "the importance of one neutron or precursor at time \( t \), before \( t_\ast \), is equal to the total importance of its probable progeny at any later time before \( t_\ast \)." In order to develop a perturbation formulation for linear functionals of the real neutron flux, Usachev (1963) made use of this concept, developed also by Kadomtsev (1957) with respect to radiation transfer problems, in relation to the steady-state case. He considered all the neutrons of the critical system as prompt, and belonging to batches of successive generations or cycles (which may be assumed to last a prompt neutron lifetime). This approximation does not have any influence on the results he derives since they are related to asymptotic quantities of the critical system. We will now recall briefly his derivation of perturbation formulae using the simple nomenclature introduced above.

Let us count neutron generations, as defined above, backward, and define at the first one a detectable process \( Q \) which represents in general a reaction rate and may be expressed then by a linear functional of the real flux; we have:

\[
Q = \int \varphi h \, dr \, d\Omega
\]  

(3)

where \( h(r, \Omega) \) is an assigned function. Considering now an importance function \( \psi^* \) in the sense defined by Lewins relative to process \( Q \), such a function will depend on time or, in Usachev's formulation, on generation index \( i \). By definition it may be written:

\[
Q = \int \psi_i^* \varphi \, dr \, d\Omega \quad (i = 1, 2, \ldots)
\]  

(4)

where \( \psi_i^* (r, \Omega) \) is the new importance function at the \( i \)-th generation. The meaning of equation (4) is clear. It represents the sum of the contributions to the quantity \( Q \) from all the neutrons belonging to the \( i \)-th generation. The evaluation of the importance function \( \psi_i^* \) is very simple since it corresponds to solving the iterative equations:

\[
\begin{align}
-\mathcal{A}^* \psi_1^* &= h \\
-\mathcal{A}^* \psi_i^* &= \mathcal{F}^* \psi_{i-1}^* 
\end{align}
\]  

(5)

As may be seen, the solution of this system corresponds to the calculation of the conventional adjoint flux \( \varphi^* \) starting from a source \( h \) and following an iteration routine. So we have:

\[
\lim_{i \to \infty} \psi_i^* = c \varphi^*.
\]  

(6)

Then Usachev considers ratios of quantities (representing in particular conversion ratios):

\[
R = \frac{Q_1}{Q_2} = \frac{\int \varphi h_1 \, dr \, d\Omega}{\int \varphi h_2 \, dr \, d\Omega}
\]  

(7)
and introduces a linear functional $Q_0$:

$$Q_0 = \int \varphi g \, dr \, d\Omega = 0$$

(8)

where $g$ is given by:

$$g = \frac{h_1}{Q_1} - \frac{h_2}{Q_2}.$$  

(9)

It is possible to define also for this functional importance functions $\psi_i^*$ given by a system of equations (5) where $h$ is now replaced by $g$. It can easily be seen that in this case the coefficient $c$ in equation (6) is equal to zero.

Let us now introduce a perturbation into the steady-state system, which will in general affect its criticality. We will then introduce a further change to restore the system to critical, for instance by slightly moving a control rod or by adding or subtracting fissile material. Such a perturbation and the changes made to restore criticality add up to a change $\delta \varphi$ in the Boltzmann operator and, in general, to a change $\delta g$ in the function $g$. As a consequence the ratio $R$ and the quantity $Q_0$ will generally change. By a proper balance of importance loss at each generation Usachev derives the perturbation expression:

$$\frac{\delta R}{R} = \delta Q_0 = \int \varphi' \delta g \, dr \, d\Omega + \int \psi^* \delta \varphi \varphi' \, dr \, d\Omega$$

(10)

where $\varphi'$ represents the perturbed flux, while

$$\delta g = \frac{\delta h_1}{Q_1} - \frac{\delta h_2}{Q_2}$$

(11)

and

$$\psi^* = \sum_{i=1}^{\infty} \psi_i^*.$$  

(12)

The first term on the right-hand side of equation (10) is related to a change, if any, of the functions $h_1$ and $h_2$. The second term is related to the change of the real neutron flux corresponding to the perturbation $\delta \varphi$ and is linear with this quantity. It has exactly the same character as the conventional perturbation expression given by the numerator on the right-hand side of equation (2), with the difference that here the adjoint flux $\varphi^*$ is replaced by the function $\psi^*$. As the sum in equation (12) converges, for practical purposes the number of terms $\psi_i^*$ will be decided upon by the convergence criterion chosen, while the flux $\varphi'$ may be replaced by the unperturbed flux $\varphi$, thus dropping a second-order term.

Let us now consider a burst of neutrons injected into (or subtracted from) a critical reactor at a given moment and corresponding to the function $\delta(r, \Omega)$, which for our purposes needs to be defined apart from a normalization factor. As the final formulae depend on asymptotic quantities, we may consider all the neutrons produced by this source as belonging to batches of prompt neutron generations, and corresponding then to fluxes $\varphi_i(r, \Omega)$ where $i$ is the generation index (now counted forward).
Let us then weigh the neutrons with the adjoint functions \( \varphi^* \). Recalling the general definition of importance given by Lewins, the quantity (here called source worth):

\[
S^* = \int \varphi^* \delta \varphi^* d\Omega = \int \varphi^* \mathcal{F} \varphi_i d\Omega \quad (i = 1, 2, \ldots)
\]  

represents the contribution to the asymptotic power level from all the neutrons introduced into (or subtracted from) the system. After a large enough number of neutron generations, the fluxes \( \varphi_i \) will converge to the fundamental mode shape \( \varphi(r, \Omega) \). So we have:

\[
\lim_{t \to \infty} \varphi_i = c' \varphi
\]

where \( c' \) is given by:

\[
c' = \frac{S^*}{\int \varphi^* \mathcal{F} \varphi d\Omega} = \frac{S^*}{I_F}.
\]

As in the previous case, let us now introduce a perturbation into the steady state system and restore it, if necessary, to criticality. These operations sum up also here to a change \( \delta \mathcal{P} \) of the Boltzmann operator and, in general, to a change \( \delta \mathcal{S} \) of the source \( \mathcal{S} \). By making a proper balance of importance gain (or loss) at each generation, it is possible to write a perturbation equation, as shown by Gandini (1966) which relates \( \delta S^* \) to \( \delta \mathcal{P} \):

\[
\delta S^* = \int \varphi^* \delta \varphi^* d\Omega + \int \delta \varphi^* \delta \varphi d\Omega
\]

\[
= \int \varphi^* \delta \varphi d\Omega + \sum_{i=1}^{\infty} \int \varphi^* \delta \mathcal{P} \varphi_i d\Omega
\]

\[
= \frac{S^*}{I_F} \sum_{i=1}^{\infty} \delta \mathcal{P} \varphi_i d\Omega
\]

where \( \varphi^* \) is the perturbed adjoint flux. The sum which appears in the above equation converges, since the perturbation \( \delta \mathcal{P} \) is not supposed to change the criticality of the system and therefore, recalling equation (14), its integral terms correspondingly decrease to zero. This is in accordance with equation (2) where \( \varphi^* \) and \( \varphi^* \) may be replaced respectively by \( \varphi \) and \( \varphi^* \). For in this case \( \delta K/K \) is zero, and we may then consider the reciprocal effect of the perturbation \( (-\delta \mathcal{P}) \) in the perturbed critical system.

Let us now consider ratios \( R^* \) of the type:

\[
R^* = \frac{S_1^*}{S_2^*} = \frac{\int \varphi^* \delta_1 d\Omega}{\int \varphi^* \delta_2 d\Omega}.
\]

Also in this case we may define a functional \( S_0^* \):

\[
S_0^* = \int \varphi^* g^* d\Omega = 0
\]
In relation to the source $g^*$ we may define again fluxes $\phi_i$ which will be obtained by the iterative solution of the equations:

$$
\begin{align*}
-\mathcal{A}\phi_1 &= g^* \\
-\mathcal{A}\phi_i &= \mathcal{F}\phi_{i-1} & (i = 2, 3, \ldots).
\end{align*}
$$

Only a simple modification of a standard real flux calculation program is necessary for evaluating the fluxes $\phi_i$.

If we introduce a perturbation into the system equation (16) becomes:

$$
\delta S_0^* = \int \phi_0^* \delta g^* \, d\Omega + \sum_{i=1}^{\infty} \int \phi_i^* \delta \mathcal{P}\phi_i \, d\Omega
$$

where

$$
\mathcal{P} = \frac{\delta A_1}{S_1^*} - \frac{\delta A_2}{S_2^*}.
$$

The last term at the last member of equation (16) vanishes because $S_0^* = 0$. For the same reason also the sum:

$$
\Psi = \sum_{i=1}^{\infty} \phi_i
$$

is now convergent. Recalling that $\delta R^*/R^*$ is equal to $\delta S_0^*$ (apart from second-order terms) we may write the perturbation formula:

$$
\frac{\delta R^*}{R^*} = \int \phi_0^* \delta g^* \, d\Omega + \int \phi_0^* \delta \mathcal{P}\Psi \, d\Omega.
$$

For practical purposes $\phi_0^*$ may be replaced by the unperturbed value $\phi_0^*$, which corresponds to dropping second-order terms, and in equation (23) the sum may be limited to a finite number of terms $\phi_i$, depending upon the convergence criterion chosen. It can be seen that the first term on the right-hand side of equation (24) corresponds to a change, if any, of functions $\phi_1$ and $\phi_2$ while the second term corresponds to the adjoint flux change produced by the perturbation $\delta \mathcal{P}$. This last term presents also the same characteristics as the numerator of the conventional perturbation expression given by equation (2) with the only difference that here the real flux has been replaced by the function $\Psi$.

3. BI-LINEAR FUNCTIONALS OF THE REAL AND ADJOINT NEUTRON FLUXES

Let us now consider bi-linear functionals $T$ of the real and adjoint neutron fluxes:

$$
T = \int \phi^* \mathcal{H}\phi \, d\Omega
$$

where $\mathcal{H}$ in general represents a matrix operator. Generally ratios of quantities of this type, relative to matrix operators $\mathcal{H}_1$ and $\mathcal{H}_2$, may represent reactivity absolute
Bi-linear functionals of the real and adjoint neutron fluxes

values, prompt neutron lifetimes, reactivity ratios. We will generally call these quantities:

$$\rho_{\text{bil}}(\mathcal{H}_1, \mathcal{H}_2) = \frac{T_1}{T_2} \int \varphi^* \mathcal{H}_1 \varphi \, d\Omega - \int \varphi^* \mathcal{H}_2 \varphi \, d\Omega. \quad (26)$$

In order to derive a complete perturbation formula relative to $\rho_{\text{bil}}$ we will introduce another functional $T_0$:

$$T_0 = \int \varphi^* \mathcal{I} \varphi \, d\Omega \quad (27)$$

where

$$\mathcal{I} = \frac{\mathcal{H}_1}{T_1} - \frac{\mathcal{H}_2}{T_2}. \quad (28)$$

In relation to the functional $T_0$ we may now consider Usachev's importance functions $\Psi_i^*$ as described in the previous section. Their values will be given by the system of equations (5) where the source $h$ will be replaced by $(\mathcal{I}^* \varphi^*)$, where $\mathcal{I}^*$ is the adjoint of operator $\mathcal{I}$. Similarly we may evaluate the fluxes $\varphi_i$ defined in the same section corresponding to a source $(\mathcal{I} \varphi)$ of neutrons which replaces $\varphi^*$ in the system of equations (20). Let us then introduce, as in the previous cases, a perturbation into the steady-state system and let us restore it if necessary to criticality. The effect of these operations corresponds generally to a change $\delta \mathcal{I}$ of the matrix $\mathcal{I}$ and to a change $\delta \mathcal{P}$ of the Boltzmann operator, which in turn affects the real and adjoint fluxes. We may write in general:

$$\delta T_0 = \int \varphi^* \delta \mathcal{I} \varphi \, d\Omega + \int \delta \varphi (\mathcal{I} \varphi^*) \, d\Omega + \int \delta \varphi^* (\mathcal{I}^* \varphi) \, d\Omega. \quad (29)$$

We have neglected here the second-order term: $\int \delta \varphi^* \mathcal{I} \delta \varphi \, d\Omega$.

We now make use of both the perturbation formulae derived previously to account for the changes induced by the perturbation into the real and adjoint fluxes in relation to the integral quantity $T_0$. Recalling that $\delta \rho_{\text{bil}}/\rho_{\text{bil}}$ is equal to $\delta T_0$ (apart from second-order terms) we obtain the general expression:

$$\frac{\delta \rho_{\text{bil}}}{\rho_{\text{bil}}} = \int \varphi^* \delta \mathcal{I} \varphi \, d\Omega + \int \delta \varphi (\mathcal{I}^* \varphi^*) \, d\Omega + \int \delta \varphi^* (\mathcal{I} \varphi) \, d\Omega \quad (30)$$

where $\varphi'$ and $\varphi^*$ represent the perturbed real and adjoint fluxes, while:

$$\delta \mathcal{I} = \frac{\delta \mathcal{H}_1}{T_1} - \frac{\delta \mathcal{H}_2}{T_2}.$$

The functions $\Psi_i^*$ and $\Psi^*$ have already been defined, and for practical purposes the fluxes $\varphi'$ and $\varphi^*$ may be replaced by the unperturbed ones $\varphi$ and $\varphi^*$. In this way all the functions appearing in equation (30) are known in terms of unperturbed quantities. On the right-hand side, the first term corresponds to a change, if any, of the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ consequent on the perturbation, the second term corresponds to the contribution of the real flux change, and the last term to the adjoint flux change. As can easily be seen, all these three terms, apart from the definition of the various vector functions, are of the type which appears in the numerator of equation (2), i.e.
they may be evaluated following a numerical procedure similar to that used for conventional perturbation theory. It should be noted also that the dropping of second-order terms made in the course of this derivation was relevant to relative changes $\delta_{\rho_{\text{rel}}}/\rho_{\text{rel}}$ which represent first-order terms. In the cases where $\rho_{\text{rel}}$ represents a reactivity worth, which may be assumed first order, these approximations amount to dropping third-order terms with respect to the reactivity of the system.

As mentioned before, all the preceding analysis has for simplicity of nomenclature been based upon multigroup theory, thus leading to a vector and matrix notation. No limitation was imposed on the number of groups, so that changing to continuous operators is a straightforward procedure. In this case the scalar products which appear in the above formulae will be replaced by integrals over the energy variable.

4. POSSIBLE APPLICATIONS

The availability of a perturbation expression relative to $\rho_{\text{rel}}(\mathcal{H}_1, \mathcal{H}_a)$ allows evaluations to be made of many integral quantities, namely: reactivity worths, reactivity ratios, prompt neutron lifetimes. In such cases attention must be paid to the exact formulation of the integral quantity considered, so that appropriate use of equation (30) can be made.

In the case of reactivity worths, $\mathcal{H}_1$ represents a change $\Delta \mathcal{P}$ of the Boltzmann matrix operator, while $\mathcal{H}_a$ is the fission source matrix $\mathcal{F}$. We will consider now two distinct types of perturbation. The first type occurs when the perturbation $\delta \mathcal{P}$, affecting $\rho_{\text{rel}}$ as shown in equation (30), represents a size effect (e.g. self-shielding and self-multiplication). In this case $\delta \mathcal{P}$ is of course proportional to $\Delta \mathcal{P}$ itself. The second type of perturbation occurs when $\delta \mathcal{P}$ of equation (30) is independent from $\Delta \mathcal{P}$. This occurs when arbitrary changes of the nuclear properties of the system are considered. Other types of perturbation may then be considered as combinations of the ones described above.

A typical example of the first type of perturbation is represented by a sample inserted into a reactor. The reactivity produced by this insertion is given exactly by equation (2). In this case we are concerned with obtaining information on the difference between this expression and that given by equation (2a) which is normally used. The relative value of this difference, dropping second-order terms, may be written:

$$D_1 = \frac{(\Delta K/K) - (\Delta K/K)_{\text{approx}}}{(\Delta K/K)}$$

$$= \int \phi^* \Delta \mathcal{P} \phi \, d\Omega + \int \Delta \phi (\mathcal{F}^* \phi^*) \, d\Omega$$

where $\Delta \phi = (\phi' - \phi)$ and:

$$\mathcal{F}^* = \frac{\Delta \mathcal{P}}{\int \phi^* \Delta \mathcal{P} \phi \, d\Omega} - \frac{\mathcal{F}}{\int \phi^* \mathcal{F} \phi \, d\Omega}$$

$$\Delta \mathcal{F}^* = -\frac{\mathcal{F}' - \mathcal{F}}{\int \phi^* \mathcal{F} \phi \, d\Omega}$$

(32)
Bi-linear functionals of the real and adjoint neutron fluxes

From equation (31) it is evident that in the relative difference $D_1$ no change is considered in the adjoint flux. Consequently $D_1$ is given only by the first and second terms on the right-hand side of equation (30), with $\delta \mathcal{P}$ set equal to $\Delta \mathcal{P}$. The quantity $\psi^*$ is related to the functional $\rho_{\text{bil}}$ which represents $(\delta K/K)_{\text{approx}}$ considered linear only with respect to the real flux. Dropping second-order terms in equation (30) by inserting $\varphi$ in place of $\varphi'$ we may then write:

$$
\frac{\Delta K}{K} = \left(\frac{\Delta K}{K}\right)_{\text{approx}} \left[1 + \int \varphi^* \Delta \mathcal{P} \varphi \, dr \, d\Omega + \int \psi^* \Delta \mathcal{P} \varphi \, dr \, d\Omega \right].
$$

It should be mentioned that, as the theory presupposes in principle that the perturbation (here $\Delta \mathcal{P}$) should maintain the criticality of the system, which is not generally the case here, it is always possible to envisage a sort of criticality reset which does not significantly alter the flux in the region where the sample has been inserted. This may be done implicitly recalling the definition of reactivity, which gives $\delta K/K$ in terms of an equivalent $\delta \rho_{\nu}$ change. Such change (with opposite sign) may be supposed not to alter the flux distribution and added implicitly to the perturbation. On the other hand one may also think of an experimental situation where a sample is inserted into a reactor system and correspondingly a control rod is moved, far enough from the sample not to distort the local flux, in order to restore the criticality. From these arguments it appears that the approximate expression (33) is still valid.

As to the second type of perturbation, which seems to have the most important applications, in general we need to know the effect of a perturbation $\delta \mathcal{P}$ on a quantity which can be represented by a functional of the form given by equation (2), where $\Delta \mathcal{P}$ represents the change of the Boltzmann operator in relation to the particular reactivity effect under investigation. As mentioned above, the perturbation $\delta \mathcal{P}$ represents an arbitrary change of the reactor nuclear parameters and therefore is independent from $\Delta \mathcal{P}$.

The relative difference obtained from equation (2) with and without the presence of the perturbation $\delta \mathcal{P}$, is given by the expression:

$$
D_2 = \frac{(\delta K/K)_{\text{pert}} - (\delta K/K)_{\text{ unp}}}{(\delta K/K)_{\text{ unp}}} \\
= \int \varphi_{\text{ unp}}^* \delta \mathcal{G} \varphi_{\text{ unp}} \, dr \, d\Omega + \int \delta \varphi' (\mathcal{G} \varphi_{\text{ unp}}) \, dr \, d\Omega \\
+ \int \delta \varphi^* (\mathcal{G} \varphi_{\text{ unp}}) \, dr \, d\Omega
$$

(34)

where:

$$
\begin{align*}
\delta \varphi' &= \varphi_{\text{ pert}} - \varphi_{\text{ unp}} \\
\delta \varphi^* &= \varphi_{\text{ pert}}^* - \varphi_{\text{ unp}}^*
\end{align*}
$$

(35)

and

$$
\delta \mathcal{G} = \frac{\delta (\Delta \mathcal{P})}{\int \varphi_{\text{ unp}}^* \Delta \mathcal{P} \varphi_{\text{ unp}} \, dr \, d\Omega} - \frac{\delta \mathcal{P}}{\int \varphi_{\text{ unp}} \mathcal{P} \varphi_{\text{ unp}} \, dr \, d\Omega}.
$$

(36)
In the last member of equation (34) second-order terms have been dropped. As may be seen, $\delta \phi'$ represents the flux change, after the insertion of $\Delta \phi$, produced by the perturbation $\delta \phi$ introduced into the system, while $\delta \phi^*$ represents the adjoint flux variation, before the $\Delta \phi$ insertion, produced by the same perturbation. If we then assume that $\delta \phi' = \delta \phi$, where $\delta \phi$ represents the difference between the perturbed and unperturbed fluxes before the $\Delta \phi$ insertion, we see that the relative difference $D_\alpha$ may well be represented by $\lambda_{P_{\text{fl}}}/P_{\text{fl}}$ as given by equation (30).

Among the most important reactivity effects which may be studied with these perturbation methods, mention should be made here of the coolant void and the Doppler coefficients in fast reactors. These quantities correspond to space-dependent variations $\Delta \phi$ of the Boltzmann operator in consequence of coolant removal or of cross-section changes due to temperature rise. An interesting application would be the evaluation of the effects of burn-up on the above-mentioned quantities due to the heavy isotope composition changes and to fission product build-up. In this case such material changes may be represented by a perturbation $\delta \phi$ which may also affect the operator $\phi$ as in the case of the Doppler coefficient of the system.

Concerning reactivity ratios, we may use arguments very similar to those mentioned above. In this case we have expressions of the type:
\[
P_{\text{fl}} = \frac{\int \varphi^* \Delta \phi_1 \varphi \, dr \, d\Omega}{\int \varphi^* \Delta \phi_2 \varphi \, dr \, d\Omega}
\] (37)

where $\Delta \phi_1$ and $\Delta \phi_2$ correspond to two different reactivity worths. The availability of equation (30) which allows a linear relationship between fractional changes of $P_{\text{fl}}$ and a given perturbation $\delta \phi$, suggests the use of these quantities for the analysis of zoned fast critical experiments where presently a difficult normalization seems necessary, as illustrated by Rush (1966). This idea stems from the fact that in this case $P_{\text{fl}}$ may represent quantities dependent mainly on the test region and relatively independent of the nuclear characteristics of the driver region, so that they may be more easily correlated to the integral parameters of the reference design.

For the effective prompt neutron lifetime, we write the well-known formula:
\[
I_{\text{eff}} = \frac{\int \varphi^* \varphi \, dr \, d\Omega}{\int \varphi^* \mathcal{V} \varphi \, dr \, d\Omega}
\] (38)

where $\mathcal{V}^{-1}$ is the inverse velocity diagonal matrix. As can easily be seen, its fractional change consequent on a perturbation $\delta \phi$ of the system may be well represented by a formula of the type given by equation (30), where account is given of both real and adjoint neutron flux changes.

Other applications which may be of some interest are the sensitivity studies applied to specific integral quantities of interest. The importance coefficients derived from these analyses may then be used in particular to correlate the measured values of such quantities (including the reactivity ratios) to the differential nuclear parameters in a consistency approach, as shown, for example, by Hemment and Pendlebury (1966).
Finally, mention should be made of the heterogeneity effects which seem to be of major concern in most integral measurements made on fast critical assemblies (see, for example, Rush). So far it has been possible to use only an infinite lattice approximation as shown, for instance, by James and Matthews (1962). Using the method presented above, spatially dependent heterogeneity effects could be analysed. This could be achieved by determining the functions $\psi^*$ and $\psi$, relative to the integral quantity $\rho_{ni}$ under study, in terms of the homogeneous system and with a proper choice of the space mesh intervals. The heterogeneous system could then be synthesized from the homogeneous one by appropriate displacement of the constituent materials, to which the operational change $\delta \rho$, to be used in equations (30), corresponds.

REFERENCES


