A. Gandini

HGPT IN THE THERMOHYDRAULIC FIELD

1. Introduction

Along with the introduction of the concept of importance and, parallel to it, along with the development of calculation methods and machines, from the early 60's a flourishing of perturbation methods, at first in the linear domain and then in the nonlinear one, have been proposed for the analysis of reactor core, shielding, nuclide evolution, thermohydraulics, as well as other fields.

The perturbation formulations in the reactor domain proposed by various authors may be subdivided into three main categories, according to the approach followed in their derivation:

1. The heuristic approach, making exclusive use of importance conservation concepts, adopted first in 1963 by Usachev and then extensively developed by Gandini (1967, 1976). It will be referred to, in the following, as heuristic generalized perturbation theory (HGPT) method.


3. The differential method, proposed by Oblow (1976) and then developed by Cacuci et al. (1980), based on a formal differentiation of the response considered.

All these methods can be shown equivalent to each other (Greenspan, 1975).

We shall discuss here the potential applications of the HGPT methodology to the analysis of thermohydraulic systems.

A first indication for the potential use of the HGPT methodology for dynamic studies in the composite neutron, precursor and multi-channel temperature field were given by Gandini (1976, 1981, 1987). Its equivalence in the thermohydraulic field to other approaches using formal differentiation techniques (Oblow, 1978, and Cacuci, 1980), were demonstrate by Marques Alvim et a. (1988) The application to some nonlinear problems making use of the COBRA Code and a code derived from it by a proper operator reversal procedure for the calculation of the importance function, demonstrated the validity of the method for sensitivity analysis (Andrade Lima et al., 1991, 1993, 1998). One mail advantage of HGPT approach consists in the direct derivation of the boundary conditions to be associated to the equations governing the importance function, so that the need of introducing the bilinear concomitant to account for them is avoided.

2. The HGPT method

In the HGPT method the importance function is uniquely defined in relation to a given system response, for example, a neutron dose, the quantity of plutonium in the core at end of cycle, the temperature of the outlet coolant.
The HGPT method was first derived in relation to the linear neutron density field. Then it was extended to other linear ones. For all these fields the equation governing the importance function was obtained directly by imposing that on average the contribution to the chosen response from a particle [a neutron, or a nuclide, or an energy carrier] introduced at a given time in a given phase space point of the system is conserved through time (importance conservation principle). Obviously such importance will result generally dependent on the time, position, and, when the case, energy and direction, of the inserted particle.

Consider a linear particle field density represented by vector $f$ (e.g., the multigroup neutron density field) and a response $Q$ of the type:

$$Q = \int_{10}^{1F} <s^+, f>_{sys} dt \equiv <s^+, f>,$$  \hspace{1cm} (2.1)

where $s^+$ is an assigned vector function and where $<., .>_{sys}$ indicate integration over the phase space. Weighting all the particles inserted into the system, let's assume a source $s$, with the corresponding importance ($f^*$) will obviously give the response itself, i.e.,

$$<<f^+, s>> = Q = <<s^+, f>>,$$ \hspace{1cm} (2.2)

which represents an important reciprocity relationship.

From the first derivations mentioned above the rules for determining the equation governing the importance function $f^*$ were learned. They imply, in relation to the equation governing $f^*$:

- change of sign of the odd derivatives,
- transposing matrix elements,
- reversing the order of operators,
- substitution of the source $s$ with $s^+$.

The first three rules will be generally called "operator reversal" rules. As mentioned, these rules have been obtained heuristically, by directly constructing the equations governing the importance function for the fields of interest, basing on conservation principles (see Appendix).

The HGPT method was then extended to any field governed by linear operators for which the rules for their reversal were known. In particular, it was extended to the derivative fields, obtained from expanding to first order, around a given starting solution, a number of important nonlinear equations, as those governing:

+ Expression (2.1) is also representative of more general responses, of the type

$$Q = <<L(f)>>,$$  \hspace{1cm} (2.3)

$L$ being a given function of $f$. In fact, if we extend $f$ to the field $\hat{f} = [f \ y]^T$, where $y = L(f)$, $Q$ reduces to the form of Eq.(1), i.e.,

$$Q = <<s^+, \hat{f}>>,$$

having set $s^+ = [0 \ 1]$. 


- the coupled neutron/nuclide field, relevant to core evolution and control problems,
- the temperature field, relevant to thermohydraulics.

**General Formulation.**

Consider a generic physical model defined by a number of parameters $p_j$ ($j=1,2,...,J$) and described by an N-component vector field $\mathbf{f}$ obeying equation

$$ m(f|p) = 0 . \tag{2.3} $$

Vector $\mathbf{f}(q,t)$ generally depends on the phase space coordinates $q$ and time $t$. Vector $\mathbf{p}$ represents the set of independent parameters $p_j$ ($j=1,2,...$) fully describing the system and entering into Eq.(2.3). Their value generally determines physical constants, initial conditions, source terms, etc. Equation (2.3) can be viewed as an equation comprising linear, as well as nonlinear, operators and is assumed to be derivable with respect to parameters $p_j$ and (in the Frechet sense$^*$) component functions $f_n$ ($n=1,2,...,N$).

Consider now a response of interest, or functional $Q$ given by Eq.(2.1). In the following, we shall look for an expression giving perturbatively the change $\delta Q$ of the response $Q$ in terms of perturbations $\delta p_j$ of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to each parameter $p_j$ will be obtained.

Expanding equation (2.3) around a reference solution gives setting $f_i/j = \frac{\partial f}{\partial p_j}$,

$$ \sum_{j=1}^{J} \delta p_j(\mathbf{H}f_{ij} + \mathbf{m}_{ij}) + \mathbf{O}_2 = 0 , \tag{2.4} $$

where $\mathbf{O}_2$ is a second, or higher order term, and where $\mathbf{m}_{ij} = \frac{\partial \mathbf{m}}{\partial p_j}$.

The operator $\mathbf{H}$ is given by the expression

\footnotesize
\* A Frechet derivative is a formal differentiation which, when applied to an expression $m$ function of a variable $f$, as a result gives a linear operator (defined as $\frac{\partial m}{\partial f}$ ). It coincides with a normal derivative when $m$ is an algebraic expression. If we indicate with a a linear operator applied to function $f$, the Frechet derivative of $(\alpha f)$ with respect to $f$ results $\alpha$. For example, if $\alpha = \text{grad}$, we shall have

$$ \frac{\partial (\alpha f )}{\partial f} = \frac{\partial (\text{grad } f )}{\partial f} = \text{grad} $$

If, instead, $\alpha = \int dx K(x)(.)$, we shall have

$$ \frac{\partial (\alpha f )}{\partial f} = \frac{\partial}{\partial f} \int dx K(x)f(x) = \int dx K(x)(.) . $$
\[ H = \begin{bmatrix}
\frac{\partial m_1}{\partial f_1} & \frac{\partial m_1}{\partial f_2} & \ldots & \frac{\partial m_1}{\partial f_N} \\
\frac{\partial m_2}{\partial f_1} & \frac{\partial m_2}{\partial f_2} & \ldots & \frac{\partial m_2}{\partial f_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial m_N}{\partial f_1} & \frac{\partial m_N}{\partial f_2} & \ldots & \frac{\partial m_N}{\partial f_N}
\end{bmatrix}, \tag{2.5}\]

where by \( \frac{\partial}{\partial f_n} \) we have indicated a Frechet derivative (a formal differentiation resulting into an operator) (Listernik and Sobolev, 1972).

Since parameters \( p_j \), and then their changes \( \delta p_j \), have been assumed to be independent from each other, it must follow

\[ Hf_j + m_j = 0, \tag{2.6} \]

which represents the (linear) equation governing the derivative functions \( f_j \). The source term \( m_j \) is here intended to account also, via appropriate delta functions, for the initial and, if the case, boundary conditions.

Consider now functional

\[ Q_j = \ll h^+, f_j \rr. \tag{2.7} \]

Introducing the importance \( f^* \) associated with field \( f_j \), if we use it as weight of the source term \( m_j \), and integrate space- and time-wise, according to the source reciprocity relationship, Eq.(2.2), the resulting quantity will be equivalent to functional \( Q_j \), i.e.,

\[ Q_j = \ll f^*, m_j \rr, \tag{2.8} \]

where the importance \( f^* \) obeys the (index-independent) equation

\[ H^* f^* + h^+ = 0, \tag{2.9} \]

\( H^* \) being obtained by reversing operator \( H \). As said above, this implies transposing matrix elements, changing sign of the odd derivatives, inverting the order of operators.

We can easily see that the sensitivities \( Q_{ij} \) (j=1,2,...,J) of the response with respect to system parameters can be written

\[ Q_{ij} \equiv \frac{dQ}{dp_j} = \ll \frac{\partial h^+}{\partial p_j}, f \rr + \ll f^*, \frac{\partial m}{\partial p_j} \rr, \tag{2.10} \]

where the first term at the right-hand side represents the so called direct term.
The overall change $\delta Q$ due to perturbations $\delta p_j$ ($j=1,2,...,J$) of system parameters can be written, to first order,

$$\delta Q = \sum_{j=1}^{J} \delta p_j \left[ << \frac{\partial h^+}{\partial p_j}, f >> + << \frac{\partial m}{\partial p_j}, f >> \right]$$

(2.11)

It may occur, in certain circumstances, that one or more components (e.g., $f_2$) of the vector field $f$ do not depend on a given space-time coordinate (e.g., $x$). These components, without altering the problem specifications and results, are interpreted as average quantities, and then replaced by an integral operator [e.g., $\frac{<\cdot>_{x}}{V_x}$] applied to the corresponding extended variable [to exemplify, function $f_2$ would be replaced by $\frac{<\tilde{f}_2>_{x}}{V_x}$, $\tilde{f}_2$ being the extended variable]. These extended variables will then be assumed to depend also on this coordinate, although only their average, or integrated values with respect to it are of interest and no further specification for them is required. This rule is referred to as "coordinate dependence complementation". Its use is required in order that a correct operation reversal is made to obtain the operator governing the importance function.

3. Application to the thermohydraulic field

The general HGPT methodology for sensitivity analysis may be also applied to the non linear thermohydraulic field. The methodology makes use of established reversal rules for the definition of the equation governing the importance function. A difficulty for determining this function is the definition of the boundary conditions to be adopted for its integration. The HGPT methodology solves this problem by introducing the boundary conditions relevant to the real function within the governing equations themselves, so that the need of introducing the bilinear concomitant is avoided. For showing the methodology, we consider a simple system, composed of two adjacent conductive (fuel) and convective (coolant) regions in cylindrical geometry. The fuel (inner) region had radius $R$, the coolant (anular) region has outer radius $R_{out}$, the overall height is $H$.

The temperatures $T_f(r,z,t)$ and $T_c(z,t)$ in the fuel and coolant regions, respectively, obey equations:

$$\frac{\partial (\gamma_f T_f)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r K_f \frac{\partial T_f}{\partial r}) + s_f$$

(3.1)

$$\frac{\partial (\gamma_c T_c)}{\partial t} = \frac{2\pi R}{A} h(T_R - T_c) - \frac{\partial (v\gamma_c T_c)}{\partial z} + s_c$$

(3.2)

$$-K \frac{\partial T_f}{\partial r} \bigg|_{r=R} = h[T_f(R) - T_c] \quad \text{(boundary condition)}$$

(3.3)

$\gamma_f$ = volume heat capacity of the fuel region (generally temperature dependent);
$\gamma_c$ = volume heat capacity of the coolant region (generally temperature dependent);
$R$ = outer radius of conductive region;
$A$ = transverse area (assumed constant) of the coolant channel;
$K_f$ = thermal conductivity (generally temperature dependent);
$h$ = heat transfer coefficient (generally temperature dependent);
$v$ = coolant speed;
\( s_r(r,z,t) \) = heat source in the conductive region. By a delta-type function, it can also accommodate initial conditions at \( t_0 \); 
\( s_c(z,t) \) = heat source in the convective region, usually given by a delta function of the type 
\[
 s_c(z,t) = \dot{s}_c(t) \delta(z) \tag{3.4}
\]
to account for the heat energy input with coolant entering at \( z=0 \). By another delta function, it can also accommodate initial conditions at \( t_0 \).

Let us define then the complementary function \( \tilde{T}_c(r,z) \). By the complementation rule, function \( \tilde{T}_c(r,z,t) \) is related to \( T_c(r,z) \) by the expression:
\[
< \frac{\tilde{T}_c(r,z,t)}{A_o} >_{r,c} = T_c(z,t) \tag{3.5}
\]
where \( A_o \) corresponds to the unperturbed value of \( A \) and
\[
< . >_{r,c} \equiv 2\pi \int_{R}^{R_{out}} (\cdot) r dr . \tag{3.6}
\]

Let us define a response in the generic form:
\[
Q = \int_0^{\infty} dt \left[ 2\pi \int_0^R \int_0^H T_f s_f^+ dz + \int_0^H T_c s_c^+ dz \right] = \int_0^{\infty} dt \left( < T_f s_f^+ >_{rz} + < T_c s_c^+ >_{rz} \right) \tag{3.7}
\]
with \( s_f^+(r,z) \) and \( s_c^+(z) \) given.

By appropriate delta functions, \( Q \) may represent values defined in specific points in space and time.

We rewrite equations (3.1), (3.2) and (3.3) in a form (inclusive of initial and boundary conditions) such not to alter the heat balance within each equation,
\[
m_f = -\frac{\partial (\gamma_f T_f)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r K \frac{\partial T_f}{\partial r} \right) + s_f(r,z,t) \\
- \left[ K \frac{\partial T_f}{\partial r} + h(T_f - \frac{< \tilde{T}_c >_{r,c}}{A_o}) \delta(r - R^-) + \gamma_f T_{f,0} \delta(t - t_0) \right] = 0 \tag{3.8}
\]
\[
m_c = -\frac{\partial (\gamma_c T_c)}{\partial t} + \frac{2\pi R}{A} h[< T_f \delta(r - R^-) >_f - \frac{< \tilde{T}_c >_{r,c}}{A_o}] \\
- \frac{\partial}{\partial z} \left( \nu \gamma_c \frac{< \tilde{T}_c >_{r,c}}{A_o} + \dot{s}_c(t) \delta(z) + T_{c,0} \gamma_c \delta(t - t_0) \right) = 0 \tag{3.9}
\]
Equations (3.8) and (3.9) may be written in the compact form
\[
m_f(T \mid p) = 0 \tag{3.10}
\]
with $\tilde{T} = \begin{bmatrix} T_r \\ T_c \end{bmatrix}$ and vector $\mathbf{p}$ representing system parameters $p_j$ ($j=1,...,J$), i.e., physical constants, geometric parameters, parameters entering into boundary and initial conditions.

The derivative function obeys equation

$$H\tilde{T}_{r,j} + s_j = 0$$

(3.11)

with

$$\tilde{T}_{r,j} = \frac{d\tilde{T}}{dp_j}, \quad s_j = \frac{\partial \mathbf{m}}{\partial p_j}$$

(3.12)

and

$$H = \frac{\partial \mathbf{m}}{\partial \mathbf{T}} \equiv \begin{bmatrix} \frac{\partial m_c}{\partial T_r} & \frac{\partial m_r}{\partial T_r} \\ \frac{\partial m_c}{\partial T_c} & \frac{\partial m_r}{\partial T_c} \end{bmatrix}.$$ 

(3.13)

We find, recalling that $T_c \equiv <\tilde{T}_c >_{r,c}$ and then $\frac{\partial h}{\partial \tilde{T}_c} = \frac{\partial h}{\partial T_c} \frac{\partial T_c}{\partial \tilde{T}_c} = \frac{\partial h}{\partial T_c} <\bullet >_{r,c}$, that the operator governing the derivative function results

$$H_T = \begin{bmatrix} \frac{\partial \tilde{\gamma}_f (\cdot)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial [K(\cdot)]}{\partial r} \right\} \\ -\delta(r-R^+) \left\{ \frac{\partial [K(\cdot)]}{\partial r} + h \right\} \\ \frac{h}{A} <\delta(r-R^-)(\cdot) >_{r,f} \\ \frac{\partial \tilde{\gamma}_c (\cdot)}{\partial t} - \frac{2\pi R}{A} \tilde{h} \frac{\partial \gamma}{\partial z} \end{bmatrix} <\bullet >_{r,c}$$

(3.14)*

with

$$\frac{\partial}{\partial T} \left( K \frac{\partial T}{\partial r} \right) = \frac{\partial K}{\partial T} \frac{\partial T}{\partial r} + K \frac{\partial}{\partial r} + K \frac{\partial}{\partial r} = \frac{\partial [K(\cdot)]}{\partial r}$$

* In performing the Frechet derivative, we have, in particular,
\[ \dot{\gamma}_x = \gamma_x + T_x \frac{\partial \gamma_x}{\partial T_x} \quad (x=f,c) \]  
\[ \dot{h} = h - (T_f - T_c) \frac{\partial h}{\partial T_c} \]  
\[ \nu \dot{\gamma}_c = \nu \gamma_c + T_c \frac{\partial (\nu \gamma_c)}{\partial T_c} \] 

The operator governing the importance \( T_f^* \) and \( T_c^* \) may be obtained by adopting the established operator reversal rules, in particular that, relevant to the thermohydraulic field (see Ref. /5/, p. 356),

\[ \text{Reversal of: } 2\pi \int_{R}^{R_{\text{out}}} (c)dr \equiv \text{<} \dot{c}, \text{r}^*_{c} \rightarrow \text{<} \dot{c}, r_{f} \equiv 2\pi \int_{0}^{R} (c)dr \]  
\[ \text{i.e.,} \; \quad \text{<} \dot{c}, r_{f} \rightarrow \text{<} \dot{c}, r_{c} \]  

and vice versa. In other words, the range of space integration relevant to \( T_f \) and \( T_c \), respectively, is maintained after the operator reversal.

\[ H^*_T = \begin{bmatrix} \{ \dot{\gamma}_f + K \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\cdot)}{\partial r} \right] \} & h \delta(r - R^-) \frac{\text{<} \dot{\gamma}, r_c^*}{A} \\ \h A & \frac{\delta(r - R^-)(\dot{c})}{A} \frac{\text{<} \dot{c}, r_{f}}{A} \end{bmatrix} \]  

The equations relevant to importance functions \( T_f^* \) and \( T_c^* \) relevant to response Q may now be written. Setting \( T_c^* \) in place of \( \frac{\text{<} \dot{T}_c^*, r_c^*}{A} \), they result:

\[ \begin{bmatrix} \{ \dot{\gamma}_f + K \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial (\cdot)}{\partial r} \right] \} & h \delta(r - R^-) \\ \h A & \frac{\delta(r - R^-)(\dot{c})}{A} \end{bmatrix} \frac{\text{<} \dot{T}_c^*, r_c^*}{A} \begin{bmatrix} T_f^* \\ T_c^* \end{bmatrix} + \begin{bmatrix} s_f^* \\ s_c^* \end{bmatrix} = 0 \] 

In the first line the equation relevant to \( T_f^* \) is given, inclusive of the boundary condition at \( R \). We may also write:
\[- \dot{\gamma}_f \frac{\partial T_f^*}{\partial t} = K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T_f^*}{\partial r} \right) + s_f^+ \]  
\hspace{1cm} (3.23)

\[- \dot{\gamma}_c \frac{\partial T_c^*}{\partial t} = \frac{2\pi R}{A} \hat{h}(T_f^* - T_c^*) + \nu \dot{\gamma}_c \frac{\partial T_c^*}{\partial z} + s_c^+ \]  
\hspace{1cm} (3.24)

\[K \frac{\partial T_f^*}{\partial r} \bigg|_{r=R} = h[T_f^*(R) - T_c^*] \quad \text{(boundary condition)} \]  
\hspace{1cm} (3.25)

The sensitivity expression relevant to a generic parameter \( p_j \) is given by expression (2.10), which in this case results

\[
\frac{dQ}{dp_j} = \int_0^{t_E} dt \left( \sum_{\text{f}} \frac{\partial s_f^+}{\partial p_j} T_f >_{rz} + \sum_{\text{c}} \frac{\partial s_c^+}{\partial p_j} T_c >_{z} \right) \\
+ \int_0^{t_E} dt < T_f^* \frac{\partial}{\partial p_j} \left( \frac{\partial \gamma_f T_f}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( rK \frac{\partial T_f}{\partial r} \right) + s_f(r, z, t) \right) >_{rz} + < T_f^* (r, z, 0) \frac{\partial}{\partial p_j} (\gamma_f T_{f.o}) >_{rz} \\
+ \int_0^{t_E} dt < T_c^* \frac{\partial}{\partial p_j} \left( - \frac{\partial (\gamma_c T_c)}{\partial t} + \frac{2\pi R}{A} h[T_f(R) - T_c] - \frac{\partial (\nu \gamma_c T_c)}{\partial z} \right) >_{z} + < T_c^* \frac{\partial}{\partial p_j} [\gamma_{c.o} T_{c.o}(z, 0)] >_{z} \\
+ \int_0^{t_E} \int_0^{T_c^*(0, t)} \frac{\partial s_c^+}{\partial p_j} (t) dt - \int_0^{t_E} \int_0^{T_f^*(R, z)} \frac{\partial}{\partial p_j} \left( \left[ K \frac{\partial T_f}{\partial r} \right]_r + h(T_f - T_c) \right) >_{z} \quad (3.26)
\]

**Static conditions**

At static conditions, the governing equations (3.1), (3.2) and (3.3) are simplified into

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( rK \frac{\partial T_f}{\partial r} \right) + s_f = 0 \quad (3.27)
\]

\[
\frac{2\pi R}{A} h(T_R - T_c) - \frac{\partial (\nu \gamma_c T_c)}{\partial z} + s_c = 0 \quad (3.28)
\]

\[-K \frac{\partial T_f}{\partial r} \bigg|_{r=R} = h[T_f(R) - T_c] \quad \text{(boundary condition)} \quad (3.29)
\]

Consider a response of interest defined as

\[
Q = 2\pi \int_0^R \int_0^H T_f^* s_f^+ dz + \int_0^H T_c^* s_c^+ dz \\
\equiv < T_f^* s_f^+ >_{rz} + < T_c^* s_c^+ >_{z} \quad (3.30)
\]
In order to obtain the sensitivity expression for this quantity in relation to parameter changes, we view this problem as a time dependent one, with the response given by the (equivalent) expression

\[
Q = \int_0^{t_1} dt \delta(t - \tilde{t}) \left[ \frac{H}{2\pi} \int_0^R \int_0^H T_f s_r^+ dz + \right] \\
\equiv \int_0^{t_1} dt \delta(t - \tilde{t}) \left( < T_f s_r^+ >_{rz} + < T_c s_c^+ >_{rz} \right)
\]

(3.31)

with \( \tilde{t} \) an arbitrary time between the (also arbitrary) interval \((0,t_F)\). Following the procedure described above for the time dependent case we easily obtain the sensitivity expression

\[
\frac{dQ}{dp_j} = \frac{\partial s_f^+}{\partial p_j} T_f >_{rz} + < \frac{\partial s_c^+}{\partial p_j} T_c >_{rz} + < \frac{\partial}{\partial p_j} \left[ \frac{1}{r} \frac{d}{dr} (rK \frac{\partial T_f}{\partial r}) + s_f (r,z,t) \right] >_{rz}
\]

\[
+ < \frac{\partial}{\partial p_j} \left[ \frac{\partial (\gamma r T_c)}{\partial t} + \frac{2\pi R}{A} h[T_f (r) - T_c] - \frac{\partial (v c T_c)}{\partial z} \right] >_{rz}
\]

\[
+ \frac{\partial}{\partial p_j} \left[ \frac{\partial s_c^+}{\partial p_j} (t) \frac{\partial}{\partial p_j} \left[ \frac{\partial T_f}{\partial r} \right] + \frac{h(T_f - T_c)}{t} \right] >_{rz}
\]

(3.32)

with

\[
\bar{T}_f^* (R,z) = \int_{-\infty}^{t} T_f^* (R,z,t) dt \quad \bar{T}_c^* (z) = \int_{-\infty}^{t} T_c^* (z,t) dt
\]

(3.33)

and \( T_f^* \) and \( T_c^* \) obeying equations

\[
- \dot{\gamma}_f \frac{\partial T_f^*}{\partial t} = K \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T_f^*}{\partial r}) + s_f^+ \delta(t - \tilde{t})
\]

(3.34)

\[
- \dot{\gamma}_c \frac{\partial T_c^*}{\partial t} = \frac{2\pi R}{A} \dot{h}(T_f^* - T_c^*) + v c \frac{\partial T_c^*}{\partial z} + s_c^+ \delta(t - \tilde{t})
\]

(3.35)

\[
K \frac{\partial T_f^*}{\partial r} \bigg|_{r=R} = h[T_f^* (R) - T_c^* ] . \quad \text{(boundary condition)}
\]

(3.36)

Integrating Eqs. (3.34) and (3.35) over \( t \) between \((\tilde{t} - \epsilon)\) and \((\tilde{t} + \epsilon)\), recalling that for the meaning of importance \( T_f^* \) and \( T_c^* \) at \((\tilde{t} + \epsilon)\) vanish and making \( \epsilon \to 0 \), we easily obtain that \( \dot{\gamma}_f T_f^* (\tilde{t}) = s_f^+ \) and \( \dot{\gamma}_c T_c^* (\tilde{t}) = s_c^+ \), respectively.
Integrating then the same Eqs. (3.34) and (3.35) over \( t \) from \(-\infty\) to \((\bar{t} + \varepsilon)\), with \( \varepsilon \) a small quantity, and recalling that, for the meaning of importance, \( T^*_T \) and \( T^*_c \) vanish at \((\bar{t} + \varepsilon)\), and that in a dissipative system as that considered* \( T^*_T \) and \( T^*_c \) also vanish for \( \bar{t} \to -\infty \), we obtain

\[
K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T^*_T}{\partial r} \right) + s^+_T = 0 \quad (3.37)
\]

\[
\frac{2\pi R}{A} h(T^*_T - T^*_c) + v_c^* \frac{\partial T^*_c}{\partial z} + s^+_c = 0 \quad (3.38)
\]

\[
K \left. \frac{\partial T^*_T}{\partial r} \right|_{r=\bar{R}} = h[\bar{T}^*_T(R) - \bar{T}^*_c] \quad \text{(boundary condition)} \quad (3.39)
\]

The derivative functions

The derivative functions may have an interest in those cases in which the number of responses is of the order, or higher, than the number of system parameters to be considered. In this case, the relevant solution is, for the generic \( j \)'th parameter \( p_j \):

\[
\frac{\partial (\hat{\gamma} T_{f/j})}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial (KT_{f/j})}{\partial r} \right) + s_{f/j} \quad (3.40)
\]

\[
\frac{\partial (\hat{\gamma} c_{c/j})}{\partial t} = \frac{2\pi R}{A} (hT_{f/j} - \hat{h}T_{c/j}) - \frac{\partial (v_c^* T_{c/j})}{\partial z} + s_{c/j} \quad (3.41)
\]

\[
- \left. \frac{\partial (KT_{f/j})}{\partial r} \right|_{r=\bar{R}} = [hT_{f/j}(R) - \hat{h}T_{c/j}] + \frac{\partial}{\partial \bar{p}_j} \left\{ \left( K \frac{\partial T_f}{\partial r} \right)_{\bar{R}} + h(T_f - T_c) \right\} \quad \text{(boundary condition)} \quad (3.42)
\]

Appendix. The importance function

In order to obtain the equation governing the importance function for the thermo-hydraulic field, we shall make use of a heuristic method based on the importance conservation principle. For our purposes, we shall first consider the system described by Eqs. (3.1), (3.2) and (3.3), assuming it linear, i.e., with all parameters temperature-independent.

We note that the importance functions introduced with the HGPT method are essentially related to the processes considered in a balance equation: particles born from a reaction, or leaking from an

* In fact, a heat unit introduced into the system at an asymptotic negative time would be dissipated and no effect would result at the finite time \( t \). Recalling the meaning of importance (in this case, the contribution of a heat unit introduced in a given point of the system to the response), its asymptotic (negative) value clearly results a vanishing quantity.
elementary volume, etc. The change of these processes, following a given system perturbation, are then weighted by the importance function for determining the change of the response to which it is associated. The use of these conservation principles allow us to write the equation governing the importance function and, therefore, to determine its governing operator. The comparison between this operator and that governing the real function (or its derivative, in case of a non linear system) allows us to learn the (reversal) rules to be applied for obtaining it.

In the case of the thermo-hydraulic field, although the temperature (an extensive quantity) generally appears as a variable, the processes of the balance equations concern rather the heat produced, or leaked, in other words, the enthalpy. So, in the thermo-hydraulic field, the importance function could be more properly associated for our purposes with this (intensive) quantity. As we shall see, the importance connected with the temperature results a function closely related with it.

The equations governing the specific enthalpies $u_f(r, z, t)$ [$\equiv \gamma_f T_f(r, z, t)$] and $u_c(z, t)$ [$\equiv \gamma_c T_c(z, t)$] in the conductive and convective regions, respectively, corresponding to the equivalent (here assumed linear) Eqs. (3.1), (3.2) and (3.3), are

\[
\frac{\partial u_f}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial (K u_f / \gamma_f)}{\partial r} \right) + s_f \tag{A.1}
\]

\[
\frac{\partial u_c}{\partial t} = 2\pi R \left[ \frac{u_f(R)}{\gamma_f} - \frac{u_c}{\gamma_c} \right] - \frac{\partial (v u_c)}{\partial z} + s_c \tag{A.2}
\]

\[
- \frac{1}{\gamma_f} \frac{\partial (K u_f)}{\partial r} \bigg|_{r=R} = h \left[ \frac{u_f(R)}{\gamma_f} - \frac{u_c}{\gamma_c} \right] \quad \text{(boundary condition)} \tag{A.3}
\]

where, for reasons which will appear clear later, we have formally shifted, without introducing any substantial problem alteration, coefficient $K$ into the gradient operator, as a coefficient of $u_f$.

Consider now the same response defined by Eq. (3.7), i.e.,

\[
Q = \left< \left| \frac{\partial u_f^*}{\partial t} \right| \right> \equiv \left< \left| \frac{s_f^+}{\gamma_f} \frac{s_c^+}{\gamma_c} \left[ u_f^* \right] \right| u_c^* \right> \tag{A.4}
\]

The importance $u_f^*(\tilde{r}, \tilde{z} | t)$ defined in the generic point $u(\tilde{r}, \tilde{z})$ at time $t$ may be defined as the contribution to the response $Q$ from a unit energy pulse, $\delta(r - \tilde{r})\delta(z - \tilde{z})$, introduced at point $(\tilde{r}, \tilde{z})$ at time $t$.

Since the real density $u_f$ obeys a diffusion-like equation, there is no difficulty to extend the results obtained in the neutron (or particle) density domain to the enthalpy one. After all, we may always interpret the heat density as corresponding to a density of heat carriers (phonons). The equation governing the importance function $u_f^*$ may then be written
\[-\gamma f \frac{\partial u^*_f}{\partial t} = \frac{K_1}{\gamma f} \frac{1}{r} \left( r \frac{\partial u^*_f}{\partial r} \right) + s^+_f. \tag{A.5}\]

We may now define the importance function $u^*_c(\tilde{z}|t)$ in the convective region. This function may in turn be defined as the contribution to the response $Q$ from a unit energy pulse, $\delta(z-\tilde{z})$, introduced at point $\tilde{z}$ at time $t$.

There is no such simple procedure to define the equation governing function $u^*_c$. Therefore, we shall derive the relevant importance balance equation directly. To this purpose, let us consider the difference of importance between a point $z$ (between 0 and $H$, this representing the height of the channel) at time $t$ (within 0 and $t_f$, this representing the interval within which the response $Q$ is defined) and a point $(z+\Delta z)$ (also within 0 and $H$) at time $(t+\Delta t)$ (also within 0 and $t_f$), where $\Delta t=\Delta z/v$. Based on the meaning of importance, it is clear that this difference is dependent on the influence on the temperature $T_c(z)$ (and, therefore, on the heat transferred) produced along $\Delta z$ in the time interval $\Delta t$ by the insertion of the one heat energy unit distributed in an elementary volume $A\delta z$ (supposing $\delta z \ll \Delta z$) around $z$. We have

\[
1 = A\delta z \delta u_c \tag{A.6}
\]

with $\delta u_c$ representing the increase of the specific enthalpy in such elementary volume. The temperature increase $\delta T_c$ then is

\[
\delta T_c = \frac{\delta u_c}{\gamma_c} = \frac{1}{\gamma_c A \delta z}. \tag{A.7}
\]

Therefore, the amount of energy not transferred into the coolant channel along $\Delta z$ during time $\Delta t$ is

\[
2\pi Rh \delta z \delta T_c \Delta t = \frac{2\pi R}{A} \frac{h}{\gamma_c} \Delta t \tag{A.8}
\]

From the importance conservation principle, we can then write the balance equation

\[
u^*_c(z,t) = (1 - \frac{2\pi R}{A} \frac{h}{\gamma_c} \Delta t)u^*_c(z + \Delta z, t + \Delta t) + \frac{2\pi R}{A} \frac{h}{\gamma_c} \Delta t u^*_f(R, \tilde{z}, \tilde{t}) + \frac{s^+_c(\tilde{z}, \tilde{t})}{\gamma_c} \Delta t \tag{A.9}
\]

The last term at right hand side (where $\tilde{z}$ and $\tilde{t}$ represent a point and a time within intervals $\Delta z$ and $\Delta t$) correspond to the direct contribution of such unit energy input during $\Delta t$ to the response $Q$ (to which the importance source $s^+_f$, beside $s^+_c$, is related). The penultimate term accounts for the heat that has not been transferred. In fact, this heat energy remains in the fuel energy (at position $r=R$) with its importance $u^*_f(R, \tilde{z}, \tilde{t})$, $\tilde{z}$ and $\tilde{t}$ again representing a point and a time within intervals $\Delta z$ and $\Delta t$. By summing and subtracting at the right side the term $u^*_c(z, t+\Delta t)$ and dividing by $\Delta t$, we easily obtain
\[
\frac{u^*_c(z, t) - u^*_c(z, t + \Delta t)}{\Delta t} = \frac{u^*_c(z + \Delta z, t + \Delta t) - u^*_c(z, t + \Delta t)}{\Delta t} + v\frac{u^*_c(z + \Delta z, t + \Delta t) - u^*_c(z, t + \Delta t)}{\Delta t} + \frac{s^+(\tilde{Z}, \tilde{t})}{\gamma_c}.
\] (A.10)

Making \( \Delta t \to 0 \), recalling that \( \Delta z = v\Delta t \), we have

\[
\frac{\partial u^*_c(z, t)}{\partial t} = \frac{2\pi R}{A} h \left[ u^*_f(R, \tilde{Z}, \tilde{t}) - u^*_c(z, t) \right] + \frac{\partial u^*_c(z, t)}{\partial z} + \frac{s^+(z, t)}{\gamma_c}.
\] (A.11)

Now we shall try to obtain this same equation directly following the HGPT rules so far known for reversing the operators governing the real functions.

We shall write equations (A.1) and (A.2) in the equivalent vector form

\[
\begin{bmatrix}
\frac{\partial}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{\gamma_f} \frac{\partial f(r)}{\partial r} \right] \\
- \delta(r - R^-) \left[ \frac{1}{\gamma_f} \frac{\partial f(r)}{\partial r} + \frac{h}{\gamma_f} \right] \\
h \frac{\delta(r - R^-)}{\gamma_f A} \bigg< \phi \bigg>_r \frac{\delta(r - R^-)}{\gamma_f A} \\
1 & 0 & 0 & - \int_0^\gamma f \gamma dT & 0 & 0 & \frac{u^*_f(r, z, t)}{A} \\
0 & 0 & 0 & - \int_0^\gamma f \gamma dT & 0 & 0 & \frac{u^*_c(r, z, t)}{A} \\
\end{bmatrix}
\begin{bmatrix}
u^*_c(r, z, t) \\
\end{bmatrix}
\begin{bmatrix}
s^+_f(r, z, t) \\
0 \\
0 \\
\end{bmatrix}
= 0
\] (A.12)
where the boundary condition, Eq.(A.3), has been accommodated within the first equation, and where we have complemented function $u_c(r,t)$ and $T_c(r,t)$ replacing it with $\tilde{u}_c(r,z,t)$ and $\tilde{T}_c(r,z,t)$, so that

$$\frac{<\tilde{u}_c(r,z,t)>_{r,c}}{A} = u_c(z,t) \quad \text{and} \quad \frac{<\tilde{T}_c(r,z,t)>_{r,c}}{A} = T_c(z,t).$$  \hspace{1cm} (A.13)

For following the same procedure of previous section, we shall adopt, besides those already known, the operator reversal rule defined by Eq.(3.20).

The two equations at bottom of the above system (A.12) define the general relationship between temperature and enthalpy. It is easy to verify that, with the operator reversal rules so far learned, the operator governing the importance functions $u^*_f$, $\tilde{u}_c^*$, $T^*_f$ and $\tilde{T}_c^*$, is

$$H^* = \begin{bmatrix} \frac{\partial}{\partial t} + \frac{K}{\gamma_f} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \cdot}{\partial r} \right) \\ -\delta(r-R^-) \left( -\frac{K}{\gamma_f} \frac{\partial}{\partial r} + h \right) \end{bmatrix} \frac{h}{\gamma_f} \delta(r-R^-) \frac{<\varepsilon>_{r_c}}{A} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The equation governing importances $u^*_f$, $u^*_c (\equiv \frac{<\tilde{u}_c^*>_{r,c}}{A})$, $T^*_f$ and $T^*_c (\equiv \frac{<\tilde{T}_c^*>_{r,c}}{A})$ reads

$$\begin{bmatrix} \frac{\partial}{\partial t} + \frac{K}{\gamma_f} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \cdot}{\partial r} \right) \\ -\delta(r-R^-) \left( -\frac{K}{\gamma_f} \frac{\partial}{\partial r} + h \right) \end{bmatrix} \frac{h}{\gamma_f} \delta(r-R^-) \frac{<\varepsilon>_{r_c}}{A} \begin{bmatrix} u^*_f(r,z,t) \\ u^*_c(r,z,t) \\ T^*_f(r,z,t) \\ T^*_c(r,z,t) \end{bmatrix} = 0$$

To note that here $T^*_f$ and $T^*_c$ are here associated with the two bottom algebraic equations, not with the two top differential ones, i.e., they are associated with the effect on the response $Q$ considered.
produced by a unit change of quantities $s^+_f$ and $s^+_c$, respectively, rather than by the insertion of a unit heat energy within the system this role here being played by importances $u^*_f$ and $u^*_c$..

Since it results that $T^*_f = \gamma_f$ and $T^*_c = \gamma_c$, the above equation may be reduced to

$$\begin{bmatrix}
\frac{\partial}{\partial t} + \frac{K}{\gamma_f} \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} \right] + -\delta(r-R^-)[-\frac{K}{\gamma_f} \frac{\partial}{\partial r} + h] \\
\frac{h}{\gamma_f} \delta(r-R^-) \\
\frac{h}{\gamma_A} <\delta(r-R^-)\rangle_{r,f} \\
\left[ \frac{\partial}{\partial t} - \frac{2\pi R}{\gamma_c A} h + v \frac{\partial}{\partial z} \right]
\end{bmatrix}
\begin{bmatrix}
u^*_f \\
u^*_c \\
s^+_f \\
s^+_c
\end{bmatrix} + \begin{bmatrix}
u^*_f \\
u^*_c \\
s^+_f \\
s^+_c
\end{bmatrix} = 0 \quad (A.21)

In the first line the equation relevant to $u^*_f$ is given, inclusive of the boundary condition at R. Multiplying the first and second equations by $\gamma_f$ and $\gamma_c$, respectively, we may also write Eq. (A.21) in the form

$$-\gamma_f \frac{\partial u^*_f}{\partial t} = K \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u^*_f}{\partial r} \right) + s^+_f \quad \text{ (A.16)}$$

$$-\gamma_f \frac{\partial u^*_c}{\partial t} = 2\pi R \frac{h[u^*_f(R) - u^*_c]}{A} + v \gamma_f \frac{\partial u^*_c}{\partial z} + s^+_c \quad \text{ (A.16)}$$

$$K \frac{\partial u^*_f}{\partial r} \bigg|_{r=R} = h[u^*_f(R) - u^*_c] \quad \text{ (boundary condition)} \quad \text{ (A.17)}$$

We see that the first equation, governing the importance $u^*_f$, corresponds exactly to that, Eq. (A.11), obtained basing on importance conservation principles. It is also interesting to see that the above equations correspond exactly with Eqs. (3.23), (3.24) and (3.25), governing $T^*_f$ and $T^*_c$ (here, equivalent to $u^*_f$ and $u^*_c$, obtained in previous section, replacing $\gamma_f$, $\gamma_c$ and $h$ with and $\gamma_f^*$, $\gamma_c^*$ and $h^*$, respectively.)
References


